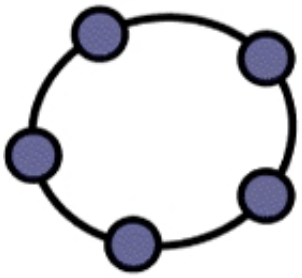

GeoGebra Workbook 6
Radian Measure, Fractions & Optimisation

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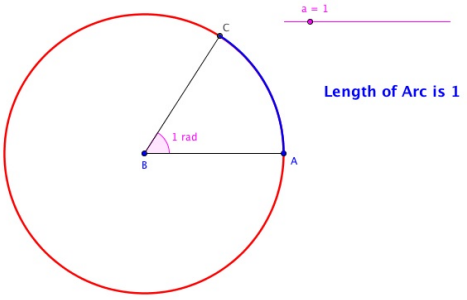
1 - Radian Measure

Preparation

- Open a new GeoGebra file.
- Hide the algebra window and the coordinate axes (View menu).
- Change the labelling setting to "New Points only" (menu Options → Labelling).
- Change the Angle measure to radians (menu Options → Angle Unit → Radians).
- You can view the file rad.html to see the completed product.



Introduction of new tools

	**Circular Arc with Center through Two Points** Hint: Click on the centre of the circle and then on the end points of the arc in an anti-clockwise direction.	*New Tool!*

Hint: Don't forget to read the toolbar help if you don't know how to use a tool. It might be helpful to try out all the new tools before you start the construction.

When we construct the Circular Arc below we will move it up a layer so that we will not see the circle underneath.

Step-by-step Instructions

1.		Create the unit circle, c by typing in the Input field `x ∧ 2 + y ∧ 2 = 1`
2.		Create the point, A by typing in the Input field `( 1 , 0 )`. Open the Properties dialog and fix this point and the circle c.
3.		Create the point, B by typing in the Input field `( 0 , 0 )`. Open the Properties dialog and fix this point.
4.		Use the Zoom tools to get the circle to a size that satisfies you.
5.		Move the Drawing Pad to position the circle correctly.
6.		Create the slider a from 0 to 2 pi with increment 0.01 and width 200.

7.		Create the point, C by typing in the Input field (`cos(a)` , `sin(a)`).
8.		Construct the angle, α, by clicking on the points A, B and C. If necessary open the Properties dialog and in the Show Label checkbox ensure that the option 'value' is chosen.
9.		Construct the line segments AB and BC.
10.		Construct the arc, e, by clicking on B, A and C in that order. Open the Properties dialog. Change the colour and line thickness. Click on the Advanced tab. In the Layer box choose 1.
11.		Type into the text box: `" Length of Arc is " + e`

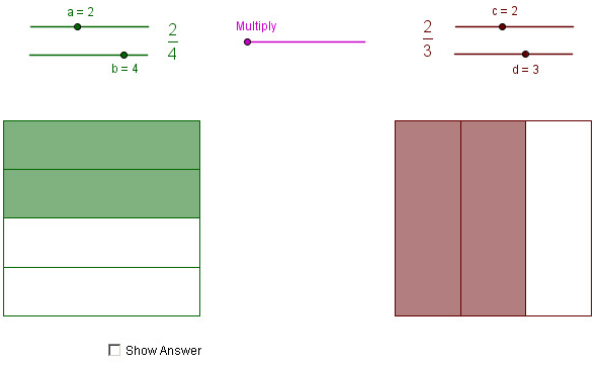
Tasks: Use the Properties dialog to enhance the appearance of the file. Create a point midway along the arc. Attach the value of the length to this point and then hide the point.
This file constructs a circle of radius 1 and thus the arc length is equal in measure to the angle. Modify the file so that the circle can be of any radius — within reason! — and then the file should be changed to show that $s = r\theta$ where the symbols have their usual meaning.

2 - Fractions — Visualising Fraction Multiplication

Many students have difficulty multiplying fractions. In this section we will present a visual approach to assist people in understanding the procedure of multiplying fractions. This section pulls together ideas and commands presented in previous workbooks - if you are unfamiliar or unclear as to what these command do then an explanation can be found in the previous workbooks.

Preparations

- Open a new GeoGebra file.
- Hide the algebra window and the coordinate axes (View menu).
- Change the labelling setting to "New Points Only"
(menu Options → Labelling).
- You can view the file fractions.html to see the completed product.



Step-by-step Instructions

1.		Create 5 sliders *a, b, c, d, e*. Allow them to vary from 0 to 5 in steps of 1. Our first fraction shall be $\frac{a}{b}$ and the second fraction shall be $\frac{c}{d}$. The slider *e* shall be used to align the two squares on top of each other
2.		Create the point A using the input field by typing the following: $\mathtt{A} = (\mathtt{0} + \mathtt{e},\ \mathtt{0})$
3.		Create the point B using the input field by typing the following: $\mathtt{B} = \mathtt{A} + (\mathtt{5},\ \mathtt{0})$
4.		Create a polygon using the regular polygon tool. Select the points A and B and type 4 into the number field that appears
5.		Create the point E using the input field by typing the following: $\mathtt{E} = (\mathtt{10} - \mathtt{e},\ \mathtt{0})$
6.		Repeat steps 3 and 4 to create a second square this time using the point E as the lower left vertex of the square
7.		Adjust the slider e. When the value of e is set to 5 the two squares should line up on top of each other
8.		Enter the following command in the input field: $\mathtt{L1} = \mathtt{Sequence}[\mathtt{A} + \mathtt{i} * (\mathtt{D} - \mathtt{A}), \mathtt{i}, \mathtt{0}, \mathtt{1}, \mathtt{1/b}]$ This creates a sequence of points located on the vertical side of the first square over the point A that splits the side into segments of length $\frac{1}{b}$ times the overall length of the side.
9.		Enter the following command in the input field: $\mathtt{L2} = \mathtt{Sequence}[\mathtt{B} + \mathtt{i} * (\mathtt{C} - \mathtt{B}), \mathtt{i}, \mathtt{0}, \mathtt{1}, \mathtt{1/b}]$
10.		Enter the following command in the input field: $\mathtt{L3} = \mathtt{Sequence}[\mathtt{Segment}[\mathtt{Element}[\mathtt{L1}, \mathtt{k}], \mathtt{Element}[\mathtt{L2}, \mathtt{k}]], \mathtt{k}, \mathtt{2}, \mathtt{b}]$ This draws a segment from the points in the list L1 to the corresponding point in the list L2
11.		Enter the following command in the input field: $\mathtt{R} = \mathtt{D} - (\mathtt{0}, \mathtt{a/b} * \mathtt{5})$
12.		Enter the following command in the input field: $\mathtt{S} = \mathtt{C} - (\mathtt{0}, \mathtt{a/b} * \mathtt{5})$
13.		Create a polygon using the points DCSR - don't forget to click on the first vertex at the end to close the polygon. This polygon is the visual representation of the fraction $\frac{a}{b}$
14.		Repeat steps 8 — 13 to create the visual representation of the fraction $\frac{c}{d}$ on the second square. This time split the square vertically rather than horizontally as you did for the first square. Turning on the Algebra View (view options) might make it easier when working with the input field

Tasks: Use the Properties dialog to enhance the appearance of the file. Hide all the unnecessary objects/labels so that only the important objects are shown.

Make the following changes/improvements to your applet:

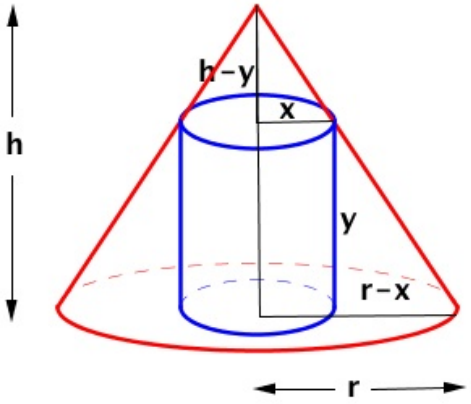
- Change it so that the first two polygons, poly1 and poly2, are no longer visible.
- Increase the slider `a` so that its value is greater than `b`. What happens? Is there a way to stop this from happening? Do the same for sliders `c` and `d`.
- Create a checkbox and use it to display the actual answer of the multiplication being carried out.

3 - Optimisation — Visualising the Maximum.

This example is very complex and gives some idea of the power of GeoGebra. We want to demonstrate the idea of the maximum value of a function. We will take a typical problem from the Calculus option in the Higher Leaving Certificate. This section involves splitting the screen and hiding the axes so that we can create our own. The problem is to find the maximum volume of a cylinder which is placed inside a cone of height h and radius r.

From the similar triangles in the diagram on the right you can see that

$$\begin{aligned} \frac{h-y}{x} &= \frac{h}{r} \\ \Rightarrow rh - ry &= hx \\ \Rightarrow y &= h - \frac{h}{r}x \end{aligned}$$



If V is the volume of the cylinder then $V = \pi x^2 y$ so we calculate the maximum volume as follows:

$$\begin{aligned} V &= \pi x^2 \left(h - \frac{h}{r}x \right) \\ &= \pi x^2 h - \frac{\pi h}{r}x^3 \\ \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}x} &= 2\pi xh - \frac{3\pi h}{r}x^2 \end{aligned}$$

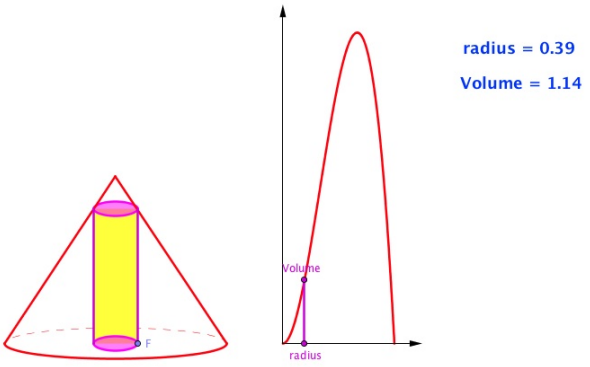
Solving $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$ yields,

$$\begin{aligned} \frac{3\pi h}{r}x &= 2\pi h \\ \Rightarrow x &= \frac{2r}{3}, \quad y = \frac{h}{3}, \quad V = \frac{4\pi h r^2}{27} \end{aligned}$$

This calculation allows you to pick appropriate values for r and h and get an idea of the scale needed for the function V.

Preparations

- Open a new GeoGebra file.
- Show the algebra window, input field and coordinate axes (View Menu).
- Change the labelling setting to "Automatic"
  (menu Options → Labelling).
- You can view the file opt.html to see the completed product.



Introduction of new tools

	**Ellipse** Hint: Select the two points that are the foci and then select a point on the ellipse.	*New Tool!*
	**Translate Object by Vector** Hint: Select object to translate then the vector.	*New Tool!*
	**Vector between Two Points** Hint: Select starting point and then end point.	*New Tool!*

Hint: Don't forget to read the toolbar help if you don't know how to use a tool. It might be helpful to try out all the new tools before you start the construction.

Step-by-step Instructions

In the following we will use H and R for the height and radius of the cone . We will use h and r for the height and radius of the cylinder inside the cone. We will create these as sliders to ensure that they appear right. At the end we will hide the sliders.

1.		Create a slider. Call it H. Let it vary between 0.1 and 5. Use the Move tool to adjust its value to 3.
2.		Create a slider. Call it R. Let it vary between 0.1 and 5. Use the Move tool to adjust its value to 2.
3.		Click on the x-axis and the y-axis to create a point at the origin. Use the Properties dialog to rename it as O.
4.		In the Input field type the equation $\texttt{V} = \texttt{Function}[\texttt{pi} * \texttt{x} \wedge 2 * \texttt{H} - \texttt{pi} * \texttt{H}/\texttt{R} * \texttt{x} \wedge 3, 0, \texttt{R}]$ This creates a graph of $V(x)$ the volume of the cylinder.

5.		In the Input field type $\mathtt{A} = (-1, 0)$ Fix this point from the Properties. It will be the bottom right of the cone.
6.		In the Input field type $\mathtt{B} = \mathtt{A} + (-2\mathtt{R}, 0)$ to create the bottom left of the cone.
7.		In the Input field create the centre of the base by typing $\mathtt{M} = (\mathtt{A} + \mathtt{B})/2$
8.		The tip of the cone is $\mathtt{C} = \mathtt{M} + (0, \mathtt{H})$
9.		Click on the points B and C to form a, the segment joining them.
10.		Click on the points C and A to form b, the segment joining them.
11.		Create the slider, c to vary between 0 and $0 \cdot 5$ in steps of $0 \cdot 01$.
12.		In the Input field type $\mathtt{D} = \mathtt{A} - (\mathtt{c}, 0)$ to create one focus of an ellipse.
13.		In the Input field type $\mathtt{E} = \mathtt{B} + (\mathtt{c}, 0)$ to create the other focus.
14.		Click on the points D, E and A to create the ellipse, d, with foci D and E passing through the point A. This will be the base of the cone. Use the slider, c, to adjust the ellipse until it looks good to you.
15.		Click on M and A to create the segment from M to A. It will be called e.
16.		Click on the segment MA to create the point F on the segment. Make sure the segment changes its appearance before you click on it.
17.		Hide the ellipse.
18.		In the Input field type $\mathtt{Arc}[\mathtt{d}, \mathtt{A}, \mathtt{B}]$ to create to create the arc, f, on the ellipse..
19.		In the Input field type $\mathtt{Arc}[\mathtt{d}, \mathtt{B}, \mathtt{A}]$ to create to create the arc, g, on the ellipse..

20.		Click on the point F and then the point M to create the point F' which is the image of F in a central symmetry in M.
21.		In the Input field type $\texttt{r} = \texttt{abs}(\texttt{x}(\texttt{M}) - \texttt{x}(\texttt{F}))$ to get r, the radius of the cylinder.
22.		In the Input field type $\texttt{h} = \texttt{H} - \texttt{H} * \texttt{r}/\texttt{R}$ to get h, the height of the cylinder.
23.		In the Input field type $\texttt{G} = \texttt{F} + (0, \texttt{h})$ to create the point G where the cylinder meets the cone.
24.		In the Input field type $\texttt{I} = \texttt{F}' + (0, \texttt{h})$ to create the point I where the cylinder meets the cone.
25.		Form the segment FG.
26.		Form the segment $F'I$.
27.		In the Input field type $\texttt{J} = \texttt{F} - (\texttt{c}, 0)$ to form the first focus of the ellipse that will form the base of the cylinder.
28.		Create the second focus by typing in the Input field $\texttt{K} = \texttt{F}' + (\texttt{c}, 0)$
29.		Click on the points J, K and F to create the ellipse, k. You may have to adjust the slider c first to see the points. After the ellipse is created adjust the slider, c, again until the result is visually pleasing.
30.		Click on the points F and G *in that order* to create the vector u going from F to G.
31.		Click on the ellipse k and on the vector u to get the image k' of the ellipse k. This is the top surface of the cylinder.
32.		Create the point L by typing in the Input field $\texttt{L} = (\texttt{R} + 1, 0)$
33.		Create the point N by typing in the Input field $\texttt{N} = (0, 4 * \texttt{pi} * \texttt{H} * \texttt{R} \wedge 2/27 + 0 \cdot 5)$
34.		Create the vectors $\vec{OL}$ and $\vec{ON}$.

35.		Use the Properties dialog to hide everything except the graph of $V(x)$, the vectors $\vec{OL}$ and $\vec{ON}$, the point F and the drawing of the cone and cylinder. If any of the visible objects are labelled hide the labels too.
36.		Create the point P by typing in the Input field $\mathtt{P} = (\mathtt{r}, 0)$ Use the Properties dialog to give it the caption `radius` and display the caption.
37		Create the point Q by typing in the Input field $\mathtt{Q} = (\mathtt{r}, \mathtt{V}(\mathtt{r}))$ Use the Properties dialog to give it the caption `Volume` and display the caption.
38.		Create the segment joining the points P and Q.
39.		Create the text $\mathtt{"radius} = \mathtt{"} + \mathtt{r}$
40.		Create the text $\mathtt{"Volume} = \mathtt{"} + \mathtt{V(r)}$

Task: Use the various tools that you have learned about to enhance the appearance of your file. Can you create points of intersection of the cylinder sides with the ellipse to draw arcs that give the illusion that the cylinder hides the back of the cone?

4 - Challenge: Surface Area

The Surface area of the cylinder can also be maximised. Try to write a GeoGebra file to show this. You may encounter some unexpected problems here which will show the need to do the mathematics first.

Tasks to Complete

- Open a new file in GeoGebra.
- Solve, mathematically, the problem of maximising the surface area of a cylinder enclosed in a cone.
- Check, in particular, the maximum and minimum values of the variables you will be using.
- Add explanations, instructions and problems for your students.