Solution Techniques

Aim

To demonstrate techniques for solving differential equations.

Learning Outcomes

At the end of this section you will be able to:

- Solve simple differential equations,
- Solve differential equations using the separation of variables technique.

Solving differential equations typically involves integration. Unfortunately, solving differential equations is usually more complicated than just writing down an integral and evaluating it. The only kind of differential equation that can be solved that way is the simplest kind of first-order differential equation that can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x).$$

The solution is then just the integral of f:

$$y = \int f(x)\ \mathrm{d}x.$$

Separable Equations

The next most simple kind of equation to solve is a so-called **separable equation**. A separable equation is one of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y).$$

We can rewrite such an equation in the equivalent differential form

$$\frac{\mathrm{d}y}{g(y)} = f(x)\ \mathrm{d}x,$$

where the variables have been *separated* on opposite sides of the equation.

We solve the separated equation by integrating both sides,

$$\int \frac{dy}{g(y)} = \int f(x)dx,$$

remembering to include a constant of integration on one side after the integrals are evaluated. Assuming both of the integrals can be evaluated, we can obtain a **general solution** to the given differential equation in the form

$$G(y) = F(x) + c.$$

The graphs of the curves $G(y) = F(x) + c$ for various values of c are called **solution curves** of the given differential equation. A single **initial condition** of the form $y = b$ when $x = a$ [or, equivalently, $y(a) = b$] will serve to single out a unique solution curve from those given by the general solution.

Example

Solve the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2y^3 \qquad y(1) = 3.$$

The differential equation becomes $\frac{\mathrm{d}y}{y^3} = x^2\ \mathrm{d}x$, so $\int \frac{\mathrm{d}y}{y^3} = \int x^2\ \mathrm{d}x$ and the general solution is

$$\frac{-1}{2y^2} = \frac{x^3}{3} + c$$

Since $y = 3$ when $x = 1$, we have $\frac{-1}{18} = \frac{1}{3} + c$ and therefore $c = -\frac{7}{18}$. Substituting this value into the above solution and solving for y, we obtain

$$y(x) = \pm\frac{3}{\sqrt{7 - 6x^3}}$$

Linear Differential Equations

A first order **linear** differential equation is one of the type

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x),$$

where $p(x)$ and $q(x)$ are given functions or constants.

We solve linear differential equations in the following way using the method of trial solutions:

Consider the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 10.$$

The first step in the solution is to find a solution of the homogeneous form of the previous equation, *i.e.* set the right hand side equal to zero and then solve the equation. The homogeneous form of the above equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0$$

Solving this using the method of separation of variables yields

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 2y \\ \frac{\mathrm{d}y}{y} &= 2\ \mathrm{d}x \\ \Rightarrow \ln y &= 2x + c \\ \Rightarrow y &= e^{2x}.e^{c} \qquad (\text{let } e^{c} = B) \\ \Rightarrow y &= Be^{2x} \end{aligned}$$

This solution is called the **complementary function**($y_{(cf)}$). We now go back and attempt to find a *trial* solution to the original equation. The original equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 10.$$

We base our initial guess of the trial solution on the *form* of the right hand side of the original function. The right hand side is a constant, *i.e* 10, so we guess a trial solution of the form $y = A$ where A is a constant. We then use this trial solution to solve the above equation.

$$\begin{aligned} y &= A \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \Rightarrow 0 - 2A &= 10 \\ \Rightarrow A &= -5 \end{aligned}$$

Therefore the solution is

$$y = -5$$

This is called the **particular integral**($y_{(pi)}$).

The full solution is the sum of the complementary solution and the particular integral. Therefore

$$\begin{aligned} y &= y_{(cf)} + y_{(pi)} \\ \Rightarrow y &= Be^{2x} - 5 \end{aligned}$$

If an initial condition had been given, *i.e* $y(0) = 7$, then it is at this stage that you would use the initial condition to fix the value of the parameter B.

Related Reading

Stewart, J. 1999. *Calculus.* 4^{th} Edition. Brooks/Cole Publishing Company.

Jacques, I. 1999. *Mathematics for Economics and Business.* 3^{rd} Edition. Prentice Hall.