Definite Integrals

Aim

To introduce the student to definite integrals.

Learning Outcomes

At the end of this section you will be able to:

- Identify the difference between an indefinite and a definite integral,
- Integrate a definite integral,
- Calculate consumer surplus and producer surplus using integration.

One of the most practical applications of integration is its use in finding the area under a curve. The formula for finding the area under the curve $y = f(x)$ between the points $x = a$ and $x = b$ is given by

$$\int_a^b f(x)dx.$$

The area under the curve is found by evaluating the integral at $x = b$ and subtracting the value of the integral at $x = a$. This may be written as

$$\int_a^b f'(x)dx = [f(x)]_a^b = f(b) - f(a)$$

$\int_a^b f(x)dx$ is called a **definite integral**. The numbers a and b are called the limits of integration, a being the lower limit and b being the upper limit. The constant of integration may be ignored when dealing with definite integrals since the two constants cancel each other out when evaluating $f(b) - f(a)$.

<u>Example 1</u>

Evaluate $\int_1^3 x^2 + 7x - 4dx$.

$$\begin{aligned}\int_1^3 x^2 + 7x - 4dx &= \left[\frac{x^3}{3} + 7\frac{x^2}{2} - 4x\right]_1^3 \\ &= \left[\frac{(3)^3}{3} + 7\frac{(3)^2}{2} - 4(3)\right] - \left[\frac{(1)^3}{3} + 7\frac{(1)^2}{2} - 4(1)\right] \\ &= \frac{86}{3}.\end{aligned}$$

Consumers'/Producers' Surplus

Consumer surplus is the difference between the price consumers are willing to pay for a good or service and the actual price. Producers surplus is the difference between what producers are willing and able to supply a good for and the price they actually receive.

Consumers' surplus and producers' surplus are calculated using the supply and demand curves. Assuming that the equilibrium price is P_0 and the equilibrium quantity is Q_0 then the formulas for consumer surplus and producer surplus are

$$CS = \int_0^{Q_0} D(Q) - P_0Q_0$$

$$PS = P_0Q_0 - \int_0^{Q_0} S(Q)$$

where $D(Q)$ represents the demand function written in terms of Q and $S(Q)$ represents the supply function written in terms of Q.

Example 2

Given the demand function $P = 30 - Q_d$ and the supply function $P = 15 + 2Q_s$ and assuming pure competition calculate the consumer's surplus and the producer's surplus.

First need to find the equilibrium price.

$$\begin{aligned} 30 - Q &= 15 + 2Q \\ 30 - 15 &= 2Q + Q \\ 15 &= 3Q \\ 5 &= Q \end{aligned}$$

Using $Q = 5$, and filling it into either the supply or the demand function we find $P = 25$. Therefore $Q_0 = 5$ and $P_0 = 25$.

Therefore the consumer's surplus is

$$\int_0^5 (30 - Q)dq - (5)(25)$$

$$\left[30Q - \frac{Q^2}{2}\right]_0^5 - 125$$

$$\left[30(5) - \frac{25}{2}\right] - 125 = \frac{25}{2} = 12.5$$

The producer's surplus is

$$5(25) - \int_0^5 (15 + 2Q)dq$$
$$125 - \left[15Q + Q^2\right]_0^5$$
$$125 - \left[15(5) + (5)^2\right]$$
$$125 - [75 + 25] = 25.$$

Related Reading

Jacques, I. 1999. *Mathematics for Economics and Business.* 3rd Edition. Prentice Hall.

Morris, O.D., P. Cooke. 1992. *Text & Tests 5.* The Celtic Press.