Isomorphic Graphs

Aim

To introduce and define the idea of isomorphic graphs.

Learning Outcomes

At the end of this section you will:

- Know what it means for two graphs to be isomorphic,
- Know how to check if two simple graphs are isomorphic,
- Know how to show that two more complex graphs are not isomorphic.

Often, two graphs may look completely different on paper, but are essentially the same from a mathematical point of view. Take for example the two graphs in Figure 1. These graphs are the same — they have the same vertices, the same edges and the same edge-to-endpoint function. If we relabel the vertices and edges of the graph in Figure 1(a) by the following mappings, the graphs would be the same:

$$f_1 : 1 \to a,\ 2 \to c,\ 3 \to b,\ 4 \to d \qquad f_2 : a_1 \to e_2,\ a_2 \to e_1$$

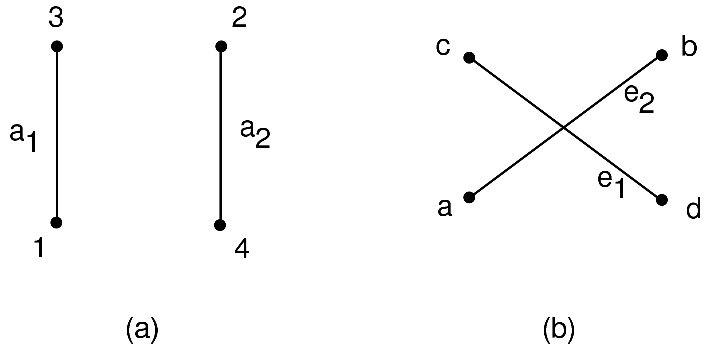

Figure 1: Isomorphic graphs

Structures that are the same except for relabelling are called *isomorphic* structures. To show that two structures are the isomorphic, we must produce a relabelling (one-to-one, onto mappings between the elements of the structures) and then show that the important properties of the structures are preserved under the relabeling.

In the case of graphs, the elements are vertices and edges. The "important property" in a graph is which edges connect which vertices.

We can use (V_1, E_1, g_1) and (V_2, E_2, g_2) to represent two graphs. V represents the vertices, E the edges and g the rule linking edges with vertices. Using this notation it is possible to define isomorphic graphs as follows:

Isomorphic Graph: Two graphs (V_1, E_1, g_1) and (V_2, E_2, g_2) are *isomorphic* if there are bijections $f_1 : V_1 \to V_2$ and $f_2 : E_1 \to E_2$ such that for each edge $a \in E_1$, $g_1(a) = x - y$ if and only if $g_2[f_2(a)] = f_1(x) - f_1(y)$.

It is not always easy to establish if 2 graphs are isomorphic or not. An exception is the case where the graphs are simple. In this case, we just need to check if there is a bijection $f : V_1 \to V_2$ which preserves adjacent vertices (i.e. if v_1, v_2 are adjacent in graph 1, then $f(v_1)$, $f(v_2)$ must be adjacent in graph 2).

If the graphs are not simple, we need more sophisticated methods to check for when two graphs are isomorphic. However, it is often straightforward to show that two graphs are *not* isomorphic. You can do this by showing *any* of the following seven conditions are true.

1. The two graphs have different numbers of vertices.
2. The two graphs have different numbers of edges.
3. One graph has parallel edges and the other does not.
4. One graph has a loop and the other does not.
5. One graph has a vertice of degree k (for example) and the other does not.
6. One graph is connected and the other is not.
7. One graph has a cycle and the other has not.

Related Reading

Gersting, J.L. 2007. *Mathematical Structures For Computer Science.* W.H. Freeman and Company.