

Planar Graphs

Aim

To introduce what a planar graph is.

Learning Outcomes

At the end of this section you will:

- Know what a planar graph is,
- Know how to use Euler's Formula to check if a graph is planar.

Planar Graph: A *planar graph* is one which can be represented (possibly after mapping it to an isomorphism) so that its edges intersect only at vertices.

Designers of integrated circuits want all components in one layer of a chip to form a planar graph so that no connections cross. The key word in the definition of a planar graph is that it *can* be drawn in a certain way.

Recall what K_4 looks like (Square with edges crossing in the center so that all vertices are adjacent). K_4 has edges which intersect at non-vertex locations. Therefore in its original state K_4 is not planar - but K_4 is isomorphic to the graph below, which is planar. Therefore K_4 is planar.

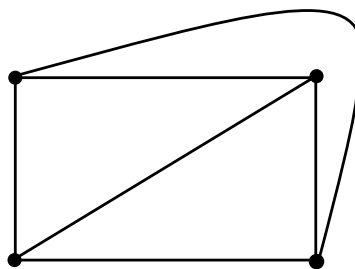


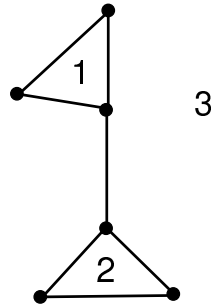
Figure 1: A K_4 planar graph

One fact about planar graphs was discovered by the Swiss mathematician Euler. A simple, connected, planar graph (when represented in its planar form, with no edges crossing) divides the plane into a number of regions, including totally enclosed regions and one infinite exterior region. Euler observed a relationship between the number n

of nodes(vertices), the number a of arcs(edges), and the number r of regions in such a graph. This relationship is known as *Euler's formula*:

$$n - a + r = 2.$$

Question: Does Euler's Formula hold true for the following graph?



Answer: We represent by the numbers 1, 2 and 3 the regions of the graph — 1 and 2 being the enclosed regions and 3 being the infinite exterior region. It is clear to see that there are 6 vertices and 7 edges in this graph. It is also easy to check that this is a simple, connected, planar graph and so Euler's formula should hold.

Euler's formula states:

$$n - a + r = 2,$$

for this graph we have

$$\begin{aligned} 6 - 7 + 3 &= 2, \\ \Rightarrow 2 &= 2. \end{aligned}$$

Therefore Euler's formula holds.

For a simple, connected, planar graph with n vertices and a edges:

1. If the planar representation divides the plane into r regions, then

$$n - a + r = 2$$

2. If $n \geq 3$, then

$$a \leq 3n - 6$$

3. If $n \geq 3$ and there are no cycles of length 3, then

$$a \leq 2n - 4$$

Related Reading

Gersting, J.L. 2007. *Mathematical Structures For Computer Science*. W.H. Freeman and Company.