Rotation Matrices

Aim

To demonstrate how to rotate a line segment using matrices.

Learning Outcomes

At the end of this section you will be able to:

- Rotate line segments located at the origin about the origin,
- Rotate line segments not located at the origin about the origin.

Rotation about the Origin

If we rotate the point with coordinates $(1, 0)$ anti-clockwise by the angle θ about the origin it moves to a point with coordinates $(\cos\theta, \sin\theta)$. This comes from basic trigonometry and can be seen in the following figure.

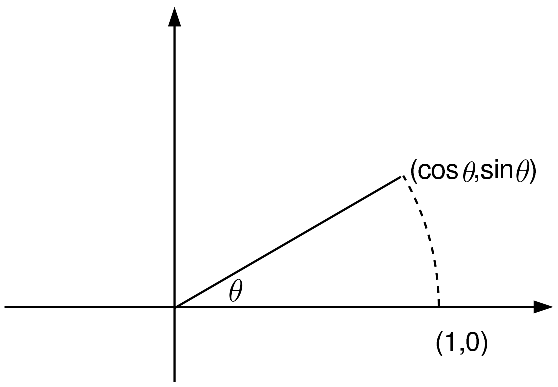

Figure 1: Rotation of a point located on the x-axis

Similarly, the point $(0, 1)$ moves to the point with coordinates $(-\sin\theta, \cos\theta)$ when rotated anti-clockwise through the angle θ.

In general, a point with "coordinates"

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

when rotated about the origin, by an angle θ, in an anti-clockwise direction results in a new point with "coordinates"

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

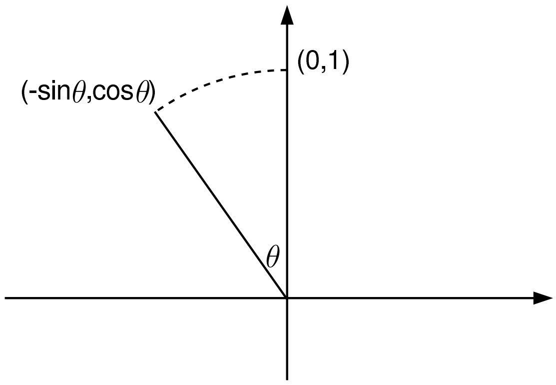

Figure 2: Rotation of a point located on the y-axis

Example

Rotate the point $(2,3)$ anti-clockwise about the origin through an angle of $\frac{\pi}{4}$.

The rotated point will be

$$\begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\cos\frac{\pi}{4} - 3\sin\frac{\pi}{4} \\ 2\sin\frac{\pi}{4} + 3\cos\frac{\pi}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 2(\frac{1}{\sqrt{2}}) - 3(\frac{1}{\sqrt{2}}) \\ 2(\frac{1}{\sqrt{2}}) + 3(\frac{1}{\sqrt{2}}) \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \end{bmatrix}$$

Rotation about a point other than the Origin

If you wish to rotate an object around a point other than the origin, then the easiest thing to do is to translate the object to the origin, rotate as normal around the origin and then undo the initial translation to reposition the object back to its new rotated location. The summary of the procedure is as follows:

Step 1: Translate the object so that the point of translation is moved to the origin.

Step 2: Rotate the relocated object as normal around the origin.

Step 3: Undo the translation in Step 1 to return the newly rotated object to its new rotated location.

Example

Find the new end points of the line segment which connects the points $(1,1)$ to $(3,3)$ when it is rotated anti-clockwise about the point $(1,1)$ through an angle of $\frac{\pi}{2}$.

First translate the line segment so that the point of rotation is moved to the origin. Therefore we want to relocate the point $(1, 1)$ to $(0, 0)$. This means that

$$P = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad P' = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$\Rightarrow T = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

From this we can work out that the line segment connecting the point $(1, 1)$ to $(3, 3)$ after translation to the origin is represented by the line segment connecting the point $(0, 0)$ to $(2, 2)$.

We now rotate the new end point $(2, 2)$ anti-clockwise about the origin through the angle $\frac{\pi}{2}$. We now multiply the rotation matrix by the point $(2, 2)$ to find the rotated new end point.

$$\begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\cos\frac{\pi}{2} - 2\sin\frac{\pi}{2} \\ 2\sin\frac{\pi}{2} + 2\cos\frac{\pi}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2(0) - 2(1) \\ 2(1) + 2(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

We now have the coordinates of the rotated line segment located at the origin. Remember that we initially translated the original line segment to the origin to make it easier to rotate it so the final step is to undo the translate from step 1. The translation matrix used in step 1 was

$$T = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

so the translation matrix to undo this is

$$T' = \begin{bmatrix} +1 \\ +1 \end{bmatrix}$$

Therefore the new end points of the line segment are

$$(0, 0) + T' = (0, 0) + (1, 1) = (1, 1),$$
$$(-2, 2) + T' = (-2, 2) + (1, 1) = (-1, 3).$$

Related Reading

Hearn, D., M. P. Baker. 1997. *Computer Graphics.* Prentice-Hall.