The Cross Product

Aim

To explain what the cross product is and to demonstrate how it works.

Learning Outcomes

At the end of this section you will be able to:

- Understand geometrically what the cross product represents,
- Calculate the determinant of two vectors,
- Calculate the cross product of any two vectors.

Determinants

Before we can calculate the cross product of two vectors we need to obtain some knowledge about determinants, which are functions that assign numerical values to square arrays of numbers. For example, if a_1, a_2, b_1 and b_2 are real numbers then we define a **2×2 determinant** by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

A **3×3 determinant** is defined in terms of 2×2 determinants by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

The right hand side of this formula is easily remembered by noting that a_1, a_2 and a_3 are the entries in the first "row" of the left hand side, and the 2×2 determinants on the right hand side arise by deleting the first row and an appropriate column (depending on which a case is being looked at) from the left hand side.

Important facts about Determinants

- If two rows in the array of a determinant are the same, then the value of the determinant is 0.
- Interchanging two rows in the array of a determinant multiplies its value by -1.

Cross Product

Definition: If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are vectors in 3-D space, then the **cross product** $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}.$$

or, equivalently,

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}.$$

Observe that the right hand side of the cross product formula has the exact same form as the right hand side of the 3×3 determinant formula. Thus we can think of the cross product as

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Note: The cross product geometrically represents a vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

Example

Let $\mathbf{u} = \langle 1, 2 - 2 \rangle$ and $\mathbf{v} = \langle 3, 0, 1 \rangle$. Find (i) $\mathbf{u} \times \mathbf{v}$ (ii) $\mathbf{v} \times \mathbf{u}$.

(i)

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \mathbf{k} = 2\mathbf{i} - 7\mathbf{j} - 6\mathbf{k}. \end{aligned}$$

(ii) Calculating $\mathbf{v} \times \mathbf{u}$ can be done easily because in this case the only difference is that row two and row three have been interchanged. We have seen earlier that interchanging the rows of a 2×2 determinant reverses its sign. Thus, by inspection

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -2\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$$

Related Reading

Anton, H., I. Bivens, S. Davis. 2005. *Calculus*. 8^{th} Edition. John Wiley & Sons.