The Dot Product

Aim

To explain what the dot product is and to demonstrate how it works.

Learning Outcomes

At the end of this section you will be able to:

- Calculate the dot product of any two vectors,
- Use the dot product to calculate the angle between two vectors.

We have previously seen how in scalar multiplication a vector is multiplied by a scalar and the result is a vector. Here we will introduce a new kind of multiplication where two vectors are multiplied by each other to produce a scalar.

Definition: If $\overrightarrow{u} = \langle u_1, u_2 \rangle$ and $\overrightarrow{v} = \langle v_1, v_2 \rangle$ are vectors in 2-D space, then the **dot product** of $\overrightarrow{u}$ and $\overrightarrow{v}$ is written $\overrightarrow{u} \cdot \overrightarrow{v}$ and is defined as

$$\overrightarrow{u} \cdot \overrightarrow{v} = u_1v_1 + u_2v_2.$$

Similarly, if $\overrightarrow{u} = \langle u_1, u_2, u_3 \rangle$ and $\overrightarrow{v} = \langle v_1, v_2, v_3 \rangle$ are vectors in 3-D space, then their dot product is defined as

$$\overrightarrow{u} \cdot \overrightarrow{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

In words, the dot product of two vectors is formed by multiplying their corresponding components and adding the resulting products. Note that the dot product of two vectors is a scalar.

Example 1

$$\begin{aligned}
\langle 3, 5 \rangle \cdot \langle -2, 4 \rangle &= 3(-2) + 5(4) = 14. \\
\langle 2, 3 \rangle \cdot \langle -3, 2 \rangle &= 2(-3) + 3(2) = 0. \\
\langle 1, 3, -4 \rangle \cdot \langle 5, -2, -3 \rangle &= 1(5) + 3(-2) + (-4)(-3) = 11.
\end{aligned}$$

Note: for the remainder of this section we shall use the bold notation to represent vectors.

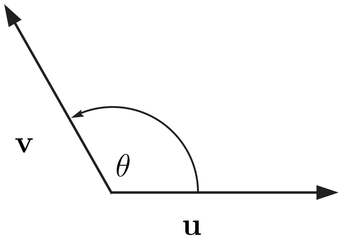
Properties of the Dot Product

If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors in 2-D or 3-D space and k is a scalar, then

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$
- $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$
- $\mathbf{0} \cdot \mathbf{v} = 0$

Angle between Vectors

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in 2-D or 3-D space that are positioned so that their initial points coincide as show below.



The angle between $\mathbf{u}$ and $\mathbf{v}$ can be defined as the smallest counterclockwise angle $(0 \leq \theta \leq \pi)$ through which one of the vectors must be rotated until it aligns with the other.

Theorem: If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in 2-D or 3-D space, and if θ is the angle between them, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\ |\mathbf{v}|}.$$

It is often convenient to express this formula as

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\ |\mathbf{v}| \cos\theta.$$

Note: $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular (because $\cos 90° = 0$).

Example 2

Find the angle between the vectors $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = -3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$.

Using the formula

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|},$$

we get

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= 1(-3)+(-2)(6)+2(2) = -11,\\ |\mathbf{u}| &= \sqrt{(1)^2+(-2)^2+2^2} = \sqrt{9} = 3,\\ |\mathbf{v}| &= \sqrt{(-3)^2+(6)^2+2^2} = \sqrt{49} = 7.\end{aligned}$$

Therefore,

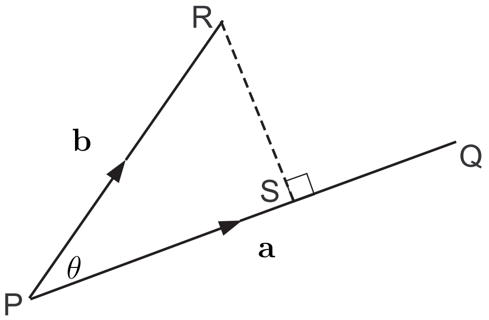
$$\cos\theta = \frac{-11}{(3)(7)} = \frac{-11}{21}.$$

Thus,

$$\theta = \cos^{-1}\left(\frac{-11}{21}\right) \approx 2.12 \text{ radians} \approx 121.6^\circ$$

Scalar and Vector Projections

One important use of dot products is in projections. The scalar projection of **b** onto **a** (also called the *component of* **b** *along* **a**) is the length of the segment PS shown in the figure below. The vector projection of **b** onto **a** (the actual vector between the points P and S) is the vector with length PS that begins at the point P and points in the same direction (or the opposite direction if the scalar projection is negative) as **a**.



The scalar projection of **b** onto **a** in the above figure is represented by the distance PS. Using trigonometry and vectors we will now come up with an expression for the scalar projection. It is obvious that

$$\cos\theta = \frac{PS}{|\mathbf{b}|}$$

We already know that

$$\mathbf{b}\cdot\mathbf{a} = |\mathbf{b}|\ |\mathbf{a}|\cos\theta$$

Therefore

$$\begin{aligned}
\mathbf{b}\cdot\mathbf{a} &= |\mathbf{b}|\ |\mathbf{a}|\cos\theta \\
\Rightarrow \mathbf{b}\cdot\mathbf{a} &= |\mathbf{b}|\ |\mathbf{a}|\frac{PS}{|\mathbf{b}|} \\
\Rightarrow \mathbf{b}\cdot\mathbf{a} &= |\mathbf{a}|PS \\
\Rightarrow \frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|} &= PS
\end{aligned}$$

Therefore the scalar projection of **b** onto **a** is given by

$$\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|}$$

The vector projection of **b** onto **a** (the actual vector between the points P and S in the previous figure) can be calculated by finding the length PS (scalar projection of **b** onto **a**) and then multiplying this value by a unit vector in the direction of **a**. Therefore the vector projection of **b** onto **a** is given by

$$\left(\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|}\right) \times \frac{\mathbf{a}}{|\mathbf{a}|}$$

Related Reading

Adams, R.A. 2003. *Calculus: A Complete Course.* 5th Edition. Pearson Education Limited.

Anton, H., I. Bivens, S. Davis. 2005. *Calculus.* 8th Edition. John Wiley & Sons.

Morris, O.D., P. Cooke. 1993. *Text & Tests 5.* The Celtic Press.