

## Gauss-Jordan Elimination

### Aim

To demonstrate the process of Gauss-Jordan elimination.

### Learning Outcomes

At the end of this section you will be able to:

- Understand the process of Gauss-Jordan elimination,
- Solve a system of linear equations using Gauss-Jordan elimination.

Consider the following matrix

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

This is an example of a matrix in **reduced row-echelon form**. To be in this form, a matrix must have the following properties:

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. (We call this the leading 1)
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else.

A matrix having properties **1**, **2** and **3** (but not necessarily **4**) is said to be in **row-echelon form**. Gaussian elimination can be used to reduce an augmented matrix to row-echelon form. A process called Gauss-Jordan elimination (which is an extended version of Gaussian elimination) is used to reduce an augmented matrix to **reduced row-echelon form**. Recall that with Gaussian elimination it was only necessary to get zeros **below** each leading 1. With Gauss-Jordan elimination it is desirable to get zeros **above** and **below** each leading 1.

Consider the following example. This is the same example as was used in the Gaussian elimination section. Gauss-Jordan elimination is the exact same as Gaussian elimination until the matrix is in row-echelon form.

Start with your augmented matrix

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{pmatrix}$$

$R_2 - 2R_1$  and  $R_3 - 3R_1$  gives

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{pmatrix}$$

$R_2 = \frac{1}{2} \times R_2$  gives

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{pmatrix}$$

$R_3 - 3R_2$  gives

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}$$

$R_3 = -2 \times R_3$  gives

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

This matrix is now in row-echelon form (upper triangular). To solve this matrix using Gauss-Jordan elimination we need to do one more step.

**Final Step:** Beginning with the last nonzero row and working upwards, add suitable multiples of each row to the rows above it to introduce zeroes above the leading 1's.

Therefore, our next step will be to add  $\frac{7}{2}$  times  $R_3$  to  $R_2$ ,

$R_2 + \frac{7}{2}R_3$  gives

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Next step is to subtract 2 times  $R_3$  from  $R_1$ ,

$$R_1 - 2R_3 \text{ gives}$$

$$\begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

We are now finished with column 3 as there are all zeros above the leading 1. The final step involves looking at column 2 and getting a zero above the leading 1. To do this we subtract  $R_2$  from  $R_1$ .

$$R_1 - R_2 \text{ gives}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\Rightarrow x = 1, y = 2 \quad \text{and} \quad z = 3.$$

This is the solution to the problem using Gauss-Jordan elimination. The final matrix is in reduced row-echelon form. You will also see that this is the same solution that was achieved using backward substitution on the row-echelon matrix that was found using Gaussian elimination.

## Related Reading

Anton, H. 1994. *Elementary Linear Algebra*. 7<sup>th</sup> Edition. John Wiley & Sons Inc.