

Inverse of a Matrix

Aim

To demonstrate how to calculate the inverse of a matrix.

Learning Outcomes

At the end of this section you will be able to:

- Calculate the inverse of a (2×2) matrix,
- Calculate the inverse of a (3×3) matrix.

The Inverse of a (2×2) Matrix

Consider the two matrices A and B . If $AB = I$, where I is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we say that B is the inverse of A .

The inverse of a square matrix A is the matrix A^{-1} which has the property

$$AA^{-1} = I = A^{-1}A \quad \text{where} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Note: If the determinant of a matrix $(ad - bc)$ is zero, then the matrix has no inverse.

Example 1

Find the inverse of the following matrix $A = \begin{pmatrix} 2 & 5 \\ -1 & 0 \end{pmatrix}$.

$$\begin{aligned} A = \begin{pmatrix} 2 & 5 \\ -1 & 0 \end{pmatrix} &\Rightarrow A^{-1} = \frac{1}{2(0) - 5(-1)} \begin{pmatrix} 0 & -5 \\ 1 & 2 \end{pmatrix}, \\ &= \frac{1}{5} \begin{pmatrix} 0 & -5 \\ 1 & 2 \end{pmatrix}, \\ &= \begin{pmatrix} 0 & -1 \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}. \end{aligned}$$

The Inverse of a (3×3) Matrix ($A|I$ Method)

To find the inverse of a (3×3) matrix we shall use row operations. This form of row operations is called Gauss-Jordan elimination because instead of only trying to reduce our matrix to upper triangular form we instead attempt to reduce it to the identity matrix (0's both above and below the leading 1's on the diagonal).

To find the inverse of a (3×3) matrix A we join the identity matrix to the right side of A as shown below,

$$(A|I)$$

and then apply row operations to this new matrix until the left side is reduced to I . These operations will convert the right side to A^{-1} so that the final matrix will have the form

$$(I|A^{-1})$$

Example 2

Find the inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 2 & 5 & 3 & | & 0 & 1 & 0 \\ 1 & 0 & 8 & | & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & -3 & | & -2 & 1 & 0 \\ 0 & -2 & 5 & | & -1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & -3 & | & -2 & 1 & 0 \\ 0 & 0 & -1 & | & -5 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & -3 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & 5 & -2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 & | & -14 & 6 & 3 \\ 0 & 1 & 0 & | & 13 & -5 & -3 \\ 0 & 0 & 1 & | & 5 & -2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & | & -40 & 16 & 9 \\ 0 & 1 & 0 & | & 13 & -5 & -3 \\ 0 & 0 & 1 & | & 5 & -2 & -1 \end{pmatrix}$$

Therefore,

$$A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

Solving Systems of Equations using Matrices

We have already seen, in previous sections, that it is possible to rewrite a system of linear equations as a matrix of the form

$$Ax = B$$

where A and B are matrices consisting of numbers and x is a matrix representing the independent variables in question. We have also seen that

$$A^{-1}A = I$$

and

$$Ix = x.$$

Using these two results it is possible to solve the above equation as follows

$$\begin{aligned} Ax &= B \\ A^{-1}Ax &= A^{-1}B && \text{premultiply both sides by } A^{-1} \\ \Rightarrow Ix &= A^{-1}B \\ \Rightarrow x &= A^{-1}B \end{aligned}$$

Therefore, the solution of a system of equations represented by $Ax = B$ is always

$$x = A^{-1}B.$$

Related Reading

Anton, H. 1994. *Elementary Linear Algebra*. 7th Edition. John Wiley & Sons Inc.

Morris, O.D., P. Cooke. 1993. *Text & Tests 5*. The Celtic Press.