Perpendicular Unit Vectors

Aim

To introduce the perpendicular unit vectors.

Learning Outcomes

At the end of this section you will be able to:

- Understand what a unit vector is,
- Write a vector in terms of the perpendicular unit vectors.

The Perpendicular Unit Vectors i, j and k

A vector of length 1 is called a **unit vector**. In an xy-coordinate system the unit vectors along the x- and y-axes are denoted by $\mathbf{i}$ and $\mathbf{j}$, respectively. In an xyz-coordinate system the unit vectors along the x-, y- and z-axes are denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, respectively. Thus,

$$\begin{aligned} \mathbf{i} &= \langle 1, 0 \rangle, \quad \mathbf{j} = \langle 0, 1 \rangle & \text{(2 − D Space)} \\ \mathbf{i} &= \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle & \text{(3 − D Space)} \end{aligned}$$

Every vector in 2-D or 3-D space is expressible uniquely in terms of the vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as follows:

$$\mathbf{v} = \langle v_1, v_2 \rangle = \langle v_1, 0 \rangle + \langle 0, v_2 \rangle = v_1 \langle 1, 0 \rangle + v_2 \langle 0, 1 \rangle = v_1 \mathbf{i} + v_2 \mathbf{j}.$$

Vectors written in terms of the unit vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ can be added together or subtracted from each other by simply carrying out the arithmetic operation on the corresponding $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ components of the vectors.

Example 1

$$(3\mathbf{i} + 4\mathbf{j}) + (-2\mathbf{i} + \mathbf{j}) = (3 - 2)\mathbf{i} + (4 + 1)\mathbf{j} = 1\mathbf{i} + 5\mathbf{j} = \mathbf{i} + 5\mathbf{j}.$$

$$5(3\mathbf{i} + 2\mathbf{j}) - (7\mathbf{i} + 3\mathbf{j}) = (15\mathbf{i} + 10\mathbf{j}) - (7\mathbf{i} + 3\mathbf{j}) = (15 - 7)\mathbf{i} + (10 - 3)\mathbf{j} = 8\mathbf{i} + 7\mathbf{j}.$$

$$|3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}| = \sqrt{3^2 + (-2)^2 + 4^2} = \sqrt{9 + 4 + 16} = \sqrt{29}.$$

Finding a Unit Vector in the same direction as a given vector

For any given vector $\mathbf{u}$, the unit vector in the direction of $\mathbf{u}$, represented by $\hat{\mathbf{u}}$, is a vector in the same direction as $\mathbf{u}$ and whose length is 1. The following formula can be used to calculate a unit vector in the direction of a given vector $\mathbf{u}$,

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|}.$$

Example 2

Find the unit vector in the same direction as $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

The vector $\mathbf{v}$ has length

$$|\mathbf{v}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3.$$

Therefore,

$$\hat{\mathbf{v}} = \frac{1}{3}\mathbf{v} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

Related Reading

Adams, R.A. 2003. *Calculus: A Complete Course.* 5^{th} Edition. Pearson Education Limited.

Anton, H., I. Bivens, S. Davis. 2005. *Calculus.* 8^{th} Edition. John Wiley & Sons.