What is a Function ?

Aim

To define what a function is.

Learning Outcomes

At the end of this section you will be able to:

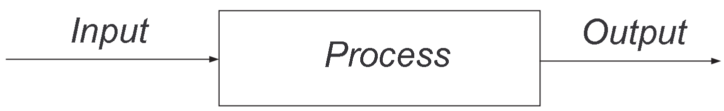
- Distinguish between a relation and a function,
- Define a function,
- Calculate the domain and range of a function.

Relations and Functions

A **relation** is just a relationship between sets of information. Think of all the people in one of your classes. Think of their heights. The pairing of names and heights is a relation. In relations and functions, the pairs of names and heights are “ordered”, which means one comes first and the other comes second. Therefore the set of ordered couples, $(name, height)$, is called a relation.

A function is a well-behaved relation. When we say that a function is “a well-behaved relation”, we mean that, given a starting point, we know exactly where to go. That is, given an x (starting point), we get only one y (finishing point).

A function can also be thought of as a system consisting of an *input*, a *process* that acts on the input and an *output* that is the result of the processing.



The processing unit of the system converts the input into the output. If the input and output of a system are numbers then provided that a single input number is processed into only one output number the process is called a **function**. A function is a rule by which an input is converted to a unique output.

Notation Used for Functions

We usually denote the input, the process and the output by letters or symbols. If the input number is labelled as x and the process labelled as f then the output is the effect of f acting on x and is accordingly labelled $f(x)$ - read as "f of x". This is the commonly accepted notation although other letters can be used to represent the input or the process.

Consider the following example. Let f be the function "add 3 to the input", let x be the input and let y be the output. In mathematical notation we write,

$$y = f(x) = x + 3,$$

This means that the function f takes the input x and produces an output $x + 3$.

We will now show how to evaluate a given function when the input x is specified.

Example 1

Evaluate the function $y = f(x) = 3x^2 + 2$ when $x = 1, 2$ and -2.

$$f(x) = 3x^2 + 2$$

$$\begin{aligned} x = 1 \quad &\Rightarrow \quad f(1) = 3(1)^2 + 2 = 3 + 2 = 5, \\ x = 2 \quad &\Rightarrow \quad f(2) = 3(2)^2 + 2 = 12 + 2 = 14, \\ x = -2 \quad &\Rightarrow \quad f(-2) = 3(-2)^2 + 2 = 12 + 2 = 14. \end{aligned}$$

Domain and Range

Definition : The set of all values that we allow the input of the function to take is called the **domain** of the function. In other words, it is all the numbers that we are allowed substitute for x into the function.

Definition : The set of values taken by the output is called the **range** of the function. In other words, these are the numbers that we get back from the function after we substitute in the x values.

If the domain is not actually specified in any particular example, it is taken to be the largest set possible.

Example 2

Find the domain and range of the following function.

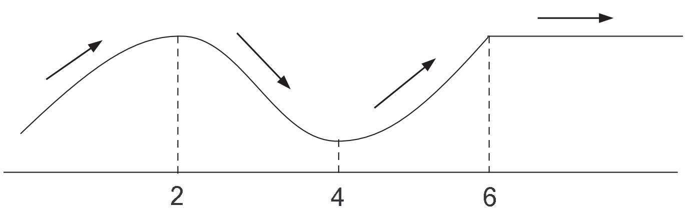
$$y = f(x) = 2x + 2, \qquad 1 \leq x \leq 3,$$

Since we are given $1 \leq x \leq 3$ the domain of the function is the interval $[1, 3]$, that is, all the values from 1 to 3 inclusively.

The range is the set of values taken by the output y. From the equation for the function it is clear to see what as x varies from 1 to 3, y varies from 4 to 8. Hence the range of the function is $[4, 8]$.

Increasing and Decreasing Functions

The terms *increasing*, *decreasing* and *constant* are used to describe the behaviour of a function over an interval as we travel left to right along its graph. The function graphed below is said to be increasing on the interval $(-\infty, 2)$, decreasing on the interval $(2, 4)$, increasing again on the interval $(4, 6)$ and constant on the interval $[6, +\infty)$.



Definition: Let f be a function defined on an interval and let x_1 and x_2 be points in that interval.

- f is **increasing** on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ for all points x_1 and x_2.
- f is **decreasing** on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ for all points x_1 and x_2.
- f is **constant** on the interval if $f(x_1) = f(x_2)$ for all points x_1 and x_2.

Vertical Line Test

A graphical test to check and see if a relation is in fact a function is the vertical line test. The test is as follows:

Given the graph of a relation, if you can draw a vertical line that crosses the graph in more than one place, then the relation is not a function.

Related Reading

Morris, O.D. 1987. *Text & Tests 1.* The Celtic Press.

Booth, D.J. 1998. *Foundation Mathematics.* 3rd Edition. Pearson Education Limited.

Mustoe, L.R, and M.D.J. Barry. 1998. *Foundation Mathematics.* John Wiley & Sons.