De Moivre's Theorem

Aim

To explain what De Moivre's Theorem is and to demonstrate its usage.

Learning Outcomes

At the end of this section you will be able to:

- Multiply two complex numbers of the form $r(\cos\theta + i\sin\theta)$,
- Understand what De Moivre's theorem is,
- Simplify complex numbers using De Moivre's Theorem.

Consider multiplying the two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

$$\begin{aligned} z_1z_2 &= r_1(\cos\theta_1 + i\sin\theta_1).r_2(\cos\theta_2 + i\sin\theta_2), \\ &= r_1r_2\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right], \\ &= r_1r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]. \end{aligned}$$

If we were to let $\theta_1 = \theta_2 = \theta$ then the above result would give,

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta.$$

From this we can deduce the following,

$$\begin{aligned} (\cos\theta + i\sin\theta)^3 &= (\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta), \\ &= (\cos 2\theta + i\sin 2\theta)(\cos\theta + i\sin\theta), \\ &= \cos 3\theta + i\sin 3\theta. \end{aligned}$$

Similarly it can be shown that

$$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta.$$

This suggests the general formula

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

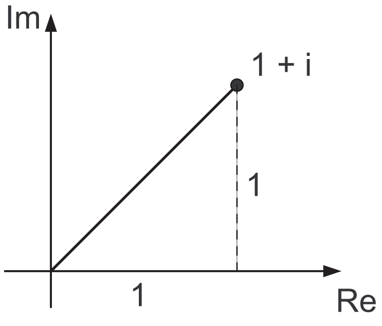
This is known as *De Moivre's Theorem.*

Example

Express $(1+i)$ in modulus-argument form and hence express $(1+i)^9$ in the form $a+ib$.

From the diagram below we can see that

$$|1+i| = \sqrt{2} \quad \text{and} \quad \tan^{-1} 1 = \frac{\pi}{4}.$$



$$\begin{aligned}
1+i &= \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]. \\
\Rightarrow (1+i)^9 &= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^9, \\
&= (\sqrt{2})^9\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right) = 2^{\frac{9}{2}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right), \\
&= 2^{\frac{9}{2}}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right), \\
&= 16\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right), \\
&= 16 + i16.
\end{aligned}$$

Related Reading

Morris, O.D., P. Cooke. 1992. *Text & Tests 4*. The Celtic Press.

Stroud, K.A. 2001. *Engineering Mathematics*. 5^{th} Edition. PALGRAVE.