What is a Complex Number ?

Aim

To explain what a complex number is and show how to represent it.

Learning Outcomes

At the end of this section you will be able to:

- Understand what a complex number is,
- Represent complex numbers on an Argand Diagram,
- Write a complex number in both the standard form and the modulus-argument form.

Up until now, you've been told that you can't take the square root of a negative number. That's because you had no numbers that, when squared, were negative. Every number was positive after you squared it. Now, however, you can take the square root of a negative number, but it involves using a new number to do it. This new number is termed an "imaginary" number because when it was originally introduced it was not considered a "real" number. We shall use the symbol i to represent an **imaginary number**. The symbol i has the property that

$$i = \sqrt{-1}.$$

Using i we can now find the square root of any negative number in the following way,

$$\sqrt{-5} = \sqrt{5. -1} = \sqrt{5}.\sqrt{-1} = \sqrt{5}i.$$

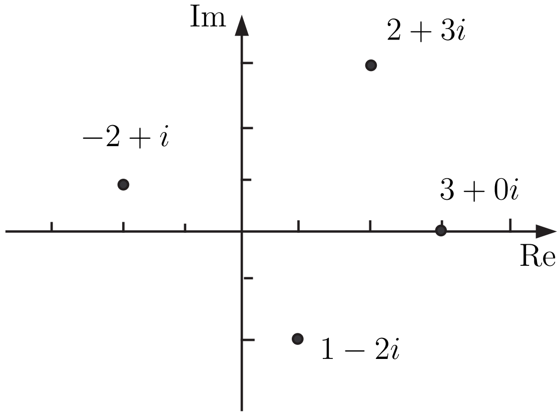
If we combine, by addition or subtraction, a real number a and an imaginary number ib, we form the number $a + ib$. A number of the form $\mathbf{a + ib}$, where a and b are real numbers, is called a **complex number**. The set of complex numbers is denoted by the letter $\mathbb{C}$.

The Argand Diagram

We have already seen how to represent an ordered pair (x, y) on a two-dimensional cartesian plane. A complex number $a + ib$ can be represented by the ordered pair (a, b)

on a special two-dimensional plane. This plane is generally referred to as the **Argand diagram** in honour of the French mathematician who first represented complex numbers in this way.

Since real numbers are represented on the x-axis and imaginary numbers on the y-axis, in the Argand diagram these axes are often referred to as the *real axis* and the *imaginary axis* respectively. The diagram below shows some complex numbers represented on the Argand diagram.

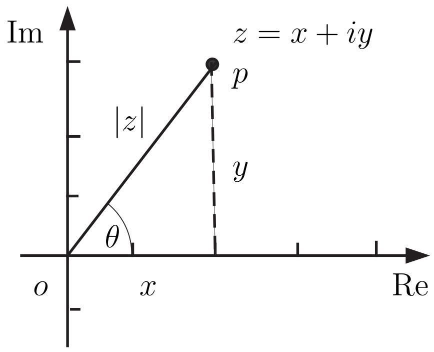


The Modulus and Argument of a Complex Number

If a point p represents the complex number $x + iy$ in the Argand Diagram then the distance from the origin o to the point p is called the **modulus** of the complex number $x + iy$. Using Pythagoras' theorem it is clear to see that the modulus of the complex number is given by

$$\text{modulus} = \sqrt{x^2 + y^2}.$$

In general the letter z is often used to represent complex numbers. If this is the case then the modulus of z is represented by $|z|$.



The angle θ between the positive real axis and the line op is called the **argument** of z.

It is often written **arg *z***. It is obvious to see that

$$\tan\theta = \frac{y}{x}.$$

If we let r represent the modulus of z in the previous diagram then it is clear to see that,

$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta,$$
$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta.$$

Therefore the complex number $x + iy$ may be written in the form

$$r\cos\theta + ri\sin\theta,$$
$$i.e. \qquad r(\cos\theta + i\sin\theta),\ \text{where } r = |z|.$$

This is called the **modulus-argument** form of a complex number. The angle θ is measured in an anti-clockwise direction from the positive real axis when θ is positive and in a clockwise direction when θ is negative. It is usual to take the value of θ in the range $-\pi < \theta \leq \pi$. This is called the **principle value** of the argument.

Related Reading

Morris, O.D., P. Cooke. 1992. *Text & Tests 4*. The Celtic Press.

Stroud, K.A. 2001. *Engineering Mathematics.* 5^{th} Edition. PALGRAVE.