Chapter 5

Dunes

The muddy colour of many rivers and the milky colour of glacial melt streams are due to the presence in the water of suspended sediments such as clay and silt. The ability of rivers to transport sediments in this way, and also (for larger particles) by rolling or saltation as bedload transport, forms an important constituent of the processes by which the Earth’s topography is formed and evolved: the science of geomorphology.

Sediment transport occurs in a variety of different (and violent) natural scenarios. Powder flow avalanches, sandstorms, lahars and pyroclastic flows are all examples of violent sediment laden flows, and the kilometres long black sandur beaches of Iceland, laid down by deposition of ash-bearing floods issuing from the front of glaciers, are testimony to the ability of fluid flows to transport colossal quantities of sediment. In this chapter we will consider some of the landforms which are built through the interaction of a fluid flow with an erodible substrate; in particular we will focus on the formation of dunes and anti-dunes in rivers, and aeolian dunes in deserts.

5.1 Patterns in rivers

There are two principal types of patterns which are seen in rivers. The first is a pattern of channel form, i.e., the shape taken by the channel as it winds through the landscape. This pattern is known as a meander, and an example is shown in figure 5.1.

The second type of pattern consists of variations in channel profile, and there are a number of variants which are observed. A distinction arises between profile variations transverse to the stream flow and those which are in the direction of flow. In the former category are bars; in the latter, dunes and anti-dunes. The formation of lateral bars results in a number of different types of river, in particular the braided and anastomosing river systems (described below).

All of these patterns are formed through an erosional instability of the uniform state when water of uniform depth and width flows down a straight channel. The instability mechanism is simply that the erosive power of the flowing water increases with water speed, which itself increases with water depth. Thus a locally deeper flow will scour its bed more rapidly, forming a positive feedback which generates
the instability. The different patterns referred to above are associated with different geometric ways in which this instability is manifested.

River meandering occurs when the instability acts on the banks. A small oscillatory perturbation to the straightness of a river causes a small secondary flow to occur transverse to the stream flow, purely for geometric reasons. This secondary flow is directed outwards (away from the centre of curvature) at the surface and inwards at the bed. As a consequence of this, and also because the stream flow is faster on the outside of a bend, there is increased erosion there, and this causes the bank to migrate away from the centre of curvature, thus causing a meander.

Figure 5.1: A meandering river. Photograph courtesy Gary Parker.

Braided rivers form because of a lateral instability which forms perturbations called bars. This is indicated schematically in figure 5.2. A deeper flow at one side of a river will cause excess erosion of the bed there, and promote the development of a

Figure 5.2: Cross section of a braided river with one lateral bar, which is exposed when the river is at low stage (i.e., the river level is low). The instability which causes the bar is operative in stormflow conditions, when the bar is submerged.

Braided rivers form because of a lateral instability which forms perturbations called bars. This is indicated schematically in figure 5.2. A deeper flow at one side of a river will cause excess erosion of the bed there, and promote the development of a
lateral bar in stormflow conditions. The counteracting (and thus stabilising) tendency is for sediments to migrate down the lateral slope thus generated. Bars commonly form in gravel bed rivers, and usually interact with the meandering tendency to form alternate bars, which form on alternate sides of the channel as the flow progresses downstream. In wider channels, more than one bar may form across the channel, and the resulting patterns are called multiple row bars. In this case the stream at low stage is split up into many winding and connected braids, and the river is referred to as a braided river, as shown in figure 5.3.


Figure 5.3: A braided river. Image from http://www.braidedriver.net.

It is fairly evident that the scouring conditions which produce lateral bars and braiding only occur during bank full discharge, when the whole channel is submerged. Such erosive events are associated with major floods, and are by their nature occasional events. In between such floods, vegetation may begin to colonise the raised bars, and if there is sufficient time, the vegetative root system can stabilise the sediment against further erosion. A further stabilising effect of vegetation is that the plants themselves increase the roughness of the bed, thus diminishing the stress trans-
mitted to the underlying sediment. If the bars become stably colonised by vegetation, then the braided channels themselves become stabilised in position, and the resulting set of channels is known as an anastomosing river system.

The final type of bedform is associated with waveforms in the direction of flow. Depending on the speed of the flow, these are called dunes or anti-dunes. At high values of the Froude number \( F > 1 \), anti-dunes occur, and at low values \( F < 1 \) dunes occur. A related feature is the ripple, which also occurs at low Froude number. Ripples are distinguished from dunes by their much smaller scale. Indeed, ripples and dunes often co-exist, with ripples forming on the larger dunes. The rest of this chapter focusses on models to describe the formation and evolution of dunes.

### 5.2 Dunes

Dunes are perhaps best known as the sand dunes of wind-blown deserts. They occur in a variety of shapes, which reflect differences in prevailing wind directions. Where wind is largely unidirectional, transverse dunes form. These are ridges which form at right angles to the prevailing wind. They have a relatively shallow upslope, a sharp crest, and a steep downslope which is at the limiting angle of friction for slip. The air flow over the dune separates at the crest, forming a separation bubble behind the dune. Transverse dunes move at speeds of metres per year in the wind direction.

Linear dunes, or seifs, form parallel to the mean prevailing wind, but are due to two different prevailing wind directions, which alternatively blow from one or other side of the dune. Such dunes propagate forward, often in a snakelike manner.

Other types of dunes are the very large star dunes (which resemble starfish), which form when winds can blow from any direction, and the crescentic barchan dunes, which occur when there is a limited supply of erodible fine sand. They take the shape of a crab-like crescent, with the arms pointing in the wind direction. Barchan dunes have been observed on Mars. (Indeed, it is easier to find images of dunes on Mars than on Earth.) Figure 5.4 shows images of the four principal types of dune described above.

As already mentioned, dunes also occur extensively in river flow. At very low flow rates, ripples form on the bed, and as the flow rate increases, these are replaced by the longer wavelength and larger amplitude dunes. These are regular scarped features, whose steep face points downstream, and which migrate slowly downstream. They form when the Froude number \( F < 1 \) (the lower régime), and are associated with river surface perturbations which are out of phase, and of smaller amplitude. The wavelength of dunes is typically comparable to the river depth, the amplitude is somewhat smaller than the depth.

When the Froude number increases further, the plane bed re-forms at \( F \approx 1 \), and then for \( F > 1 \), we obtain the upper régime, wherein anti-dunes occur. Whereas dunes are analogous to shock waves, anti-dunes are typically sinusoidal, and are in phase with the surface perturbations, which can be quite large. They may travel either upstream or (more rarely) downstream. Indeed, for the more rapid flows, backward breaking shocks occur at the surface, and chute and pool sequences form.
Anti-dunes can be found on rapid outlet streams on beaches; for example I have seen them on beach streams in Normandy and Ireland, where the velocity is on the order of a metre per second, and the flow depth may be several centimetres. A common
observed feature of such flows is their time dependence: anti-dunes form, then migrate upstream as they steepen, leading to hydraulic jumps and collapse of the pattern, only for it to re-form elsewhere. An example of such anti-dunes is shown in figure 5.5. The succession of bedforms as the Froude number increases is illustrated in figure 5.6. Anti-dunes do not form in deserts simply because the Froude number is never high enough.\textsuperscript{1}

Dunes and anti-dunes clearly form through the erosion of the underlying bed, and thus mathematical models to explain them must couple the river flow mechanics with those of sediment transport. Sediment transport models are described below. There are two main classes of bedform models. The most simple and appealing is to combine the St. Venant equations with an equation for bedform erosion. There are two ways in which sediment transport occurs, as bedload or as suspended load. Each transport mechanism gives a different model, and we shall find that a suspended load transport model can predict the instability which forms anti-dunes, but not dunes, which indeed may occur in the absence of suspended sediment transport.\textsuperscript{2} On the other hand, the St. Venant equations coupled with a simple model of bedload transport cannot predict instability, although such a model can explain the shape and speed of dunes.

The other class of model which has been used describes the variation of stream velocity with depth explicitly. One version employs potential theory, as is customarily

\textsuperscript{1}The Froude number corresponding to a wind of 20 m s\textsuperscript{-1} = 45 miles per hour over a boundary layer depth of 1 km is 0.2.

\textsuperscript{2}This also seems to be true of anti-dunes.
done in linearised surface wave theory. At first sight, this appears implausible insofar as the flow is turbulent, and indeed the model can then only explain dunes when the bed stress is artificially phase shifted. In order to deal with this properly, it is necessary to include a more sophisticated description of turbulent flow, and this can be done using an eddy viscosity model, which is then able to explain dune formation. The issue of analysing the model beyond the linear instability régime is more difficult, and some progress in this direction is described in this chapter. In Appendix B, we discuss the use of an eddy viscosity in simple models of turbulent shear flows.

5.2.1 Sediment transport

Transport of grains of a cohesionless bed occurs as bedload or in suspension. At a given flow rate, the larger particles will roll along the bed, while the smaller ones are lifted by turbulent eddies into the flow. Clearly there is a transition between the two modes of transport: saltating grains essentially bounce along the bed.

Relations to describe sediment transport are ultimately empirical, though theory suggests the use of appropriate dimensionless groups. The basic quantity is the Shields
stress, defined as the dimensionless quantity

\[ \tau^* = \frac{\tau}{\Delta \rho g D_s}. \]  \quad (5.1)

Here \( \tau \) is the basal shear stress, \( \Delta \rho = \rho_s - \rho_w \) is the excess density of solid grains over water (\( \rho_s \) is the density of the solid grains, \( \rho_w \) is the density of water), \( g \) is gravity, and \( D_s \) is the grain size. In general, grain sizes are distributed, and the Shields stress depends on the particle size. The shear stress \( \tau \) at the bed is usually related to the mean flow velocity \( u \) by the semi-empirical relation (4.9), i.e.,

\[ \tau = f \rho_w u^2, \]  \quad (5.2)

where \( f \) is a dimensionless friction factor, of typical value 0.01–0.1. ( Larger values correspond to rougher channels.)

Shields found that sediment transport occurred if \( \tau^* \) was greater than a critical value \( \tau^*_c \), which itself depends on flow rate via the particle Reynolds number

\[ Re_p = \frac{u_s D_s}{\nu}; \]  \quad (5.3)

(The friction velocity is defined to be

\[ u_s = (\tau/\rho_w)^{1/2}. \]  \quad (5.4)

Figure 5.7 shows the variation of \( \tau^*_c \) with \( u_s D_s/\nu \); except at low flow rates, \( \tau^*_c \approx 0.06 \).

### 5.2.2 Bedload

Various recipes have been given for bedload transport, that due to Meyer-Peter and Müller being popular:

\[ q^* = K[\tau^* - \tau^*_c]^3/2, \]  \quad (5.5)


where \([x]_+ = \max(x, 0)\). Here \(K = 8\), \(\tau_c^* = 0.047\), and \(q^*\) is the dimensionless bedload transport rate, defined by

\[
q^* = \frac{q_b}{(\Delta \rho g D_s^3 / \rho_w)^{1/2}},
\]

\(q_b\) being the bedload measured as volume per unit stream width per unit time.

### 5.2.3 Suspended sediment

Suspended sediment transport is effected through a balance between an erosion flux \(v_E\) and a deposition flux \(v_D\), each having units of velocity. The meaning of these is that \(\rho_s v_E\) is the mass of sediment eroded from the bed per unit area per unit time, while \(\rho_s v_D\) is the mass deposited per unit area per unit time.

**Erosion**

It is convenient to define a dimensionless erosion rate \(E\) via

\[
v_E = v_s E,
\]

where \(v_s\) is the particle settling velocity, given by Stokes’s formula

\[
v_s = \frac{\Delta \rho g D_s^2}{18 \eta},
\]

\(\eta\) being the dynamic viscosity of water. Various expressions for \(E\) have been suggested. They share the feature that \(E\) is a concave increasing function of basal stress. Typical is Van Rijn’s relationship

\[
E \propto (\tau^* - \tau_c^*)^{3/2} R_{\epsilon_T}^{1/5};
\]

typical measured values of \(E\) are in the range \(10^{-3} - 10^{-1}\).

**Deposition**

The calculation of deposition flux \(v_D\) is more complicated, as it is analogous to the calculation of basal shear stress in terms of mean velocity via an eddy viscosity model, as indicated in Appendix B. We can define the dimensionless deposition flux \(D\) by writing

\[
\rho_s v_D = v_s \bar{c} D,
\]

where \(\bar{c}\) is the mean column concentration of suspended sediment, measured as mass per unit volume of liquid, and \(D\) depends on a modified Rouse number \(R = v_s / \varepsilon_T \bar{u}\). (Here \(\varepsilon_T\) is related to the eddy viscosity; specifically \(\varepsilon_T^{-1}\) is the Reynolds number based on the eddy viscosity (see (B.9)), so the Rouse number is a Reynolds number based on particle fall velocity and eddy viscosity.) \(D\) increases with \(R\), with \(D(0) = 1\), and a typical form for \(D\) is

\[
D = \frac{R}{1 - e^{-R}}
\]

(see Appendix B for more detail).
5.3 The potential model

The first model to explain dune formation dates from 1963, and invoked a potential flow for the fluid, which was assumed inviscid and irrotational. This is somewhat at odds with the fact that it is the basal stress of the fluid which drives sediment transport, but one can rationalise this by supposing that the stress is manifested through a basal turbulent boundary layer. We restrict our attention to two dimensional motion in the \((x, z)\) plane: \(x\) is distance downstream, \(z\) is vertically upwards. The bed is at \(z = s(x, t)\), the free water surface is at \(z = \eta(x, t)\), so that the depth \(h\) is given by

\[
h = \eta - s; \quad (5.12)
\]

the geometry is shown in figure 5.8. In the potential flow model, the usual equations for the fluid flow potential \(\phi\) apply:

\[
\begin{align*}
\nabla^2 \phi & = 0 \quad \text{in} \quad s < z < \eta, \\
\phi_z & = \eta_t + \phi_x \eta_x \quad \text{on} \quad z = \eta, \\
\phi_t + g\eta + \frac{1}{2}|\nabla \phi|^2 & = \text{constant} \quad \text{on} \quad z = \eta, \\
\phi_z & = s_t + \phi_s s_x \quad \text{on} \quad z = s. \quad (5.13)
\end{align*}
\]

![Geometry of the problem](image.png)

Figure 5.8: Geometry of the problem.

The extra equation required to describe the evolution of \(s\) is the Exner equation:

\[
(1 - n) \frac{\partial s}{\partial t} + \frac{\partial q_b}{\partial x} = 0, \quad (5.14)
\]

where \(n\) is the porosity of the bed; this assumes bedload transport only, and we may take (see equations (5.5) and (5.2)) \(q_b = q_b(u)\), where \(q_b(u) > 0\). Implicitly, we suppose...
a (turbulent) boundary layer at the bed, wherein the basal stress develops through a shear layer; the basal shear stress will then depend on the outer flow velocity. We define

$$ q = \frac{q_b}{1 - n},$$  \hspace{1cm} (5.15)

so that

$$ \frac{\partial q}{\partial t} + \frac{\partial q}{\partial x} = 0. \hspace{1cm} (5.16)$$

In the absence of any dynamic effect of the bed shape on the flow, we would expect $u$, and thus also $q$, to increase as $s$ increases, due to the constriction of the flow. If indeed $q$ is an increasing function of the local bed elevation $s$, then it is easy to see from (5.16) that perturbations to the uniform state $s = 0$ will persist as forward travelling waves, and if $q$ is convex ($q''(s) > 0$) then the waves will break forwards. We interpret slip faces as the consequent shocks, so that this is consistent with observations. However, such a simple model does not allow for instability.

A simple way in which instability can be induced in the model is by allowing the maximum stress to occur upstream of the bed elevation maximum, as is indeed indicated by numerical simulations of the flow. One way to do this is to take

$$ q = q \left( u \big|_{x-\delta} \right), \hspace{1cm} (5.17)$$

that is to say, the horizontal velocity $u = \phi_x$ is evaluated at $x - \delta$ and $z = s$, where the phase lag $\delta$ is included to model the notion that in shear flow over a boundary, such a lag is indeed present. Of course (5.17) is a crude and possibly dangerous way to model this effect.

To examine the linear stability of a uniform steady state we write $s = 0, \eta = h$, \hspace{1cm} (5.18)

$$ \phi = Ux + \Phi, \quad q = q(U) + Q, \quad \eta = h + \zeta,$$

and then linearise the equations and boundary conditions (which are applied at the unperturbed boundaries $z = 0$ and $z = h$) to obtain

\begin{align*}
\nabla^2 \Phi &= 0 \text{ in } 0 < z < h; \\
\Phi_z = \zeta_t + U \zeta_x, \quad \Phi_t + g \zeta + U \Phi_x &= 0 \text{ on } z = h; \\
\Phi_z = s_t + U s_x, \quad s_t + Q_x &= 0 \text{ on } z = 0, \hspace{1cm} (5.19)
\end{align*}

where

$$ Q = q'(U) \Phi_x \big|_{x-\delta, z=0}. \hspace{1cm} (5.20)$$

For a mode of wavenumber $k$, we put

$$ (\zeta, s, Q) = (\bar{\zeta}, \bar{s}, \bar{Q}) \times e^{ikx+\sigma t}, \hspace{1cm} (5.21)$$

and write

$$ \Phi = e^{ikx+\sigma t} [A \cosh kz + B \sinh kz], \hspace{1cm} (5.22)$$
so that the boundary conditions together with (5.20) become

\[ k[A \sinh kh + B \cosh kh] = (\sigma + ikU)\tilde{\zeta}, \]

\[ (\sigma + ikU)[A \cosh kh + B \sinh kh] + g\tilde{\zeta} = 0, \]

\[ kB = (\sigma + ikU)\bar{s}, \]

\[ \sigma \bar{s} + ikQ = 0, \]

\[ Q = q'ie^{-ik\delta} A. \]  

(5.23)

Some straightforward algebra leads to

\[ \sigma[(\sigma + ikU)^2 + gk \tanh kh] + (\sigma + ikU)kq'ie^{-ik\delta}[(\sigma + ikU)^2 \tanh kh + gk] = 0, \]  

(5.24)

a cubic for \( \sigma(k) \).

Solution of this is facilitated by the observation that we can expect two modes to correspond to upstream and downstream water wave propagation, while the third corresponding to erosion of the bed may be much smaller, basically if \( q_b \) is sufficiently small. Specifically, let us assume (realistically) that \( q \ll hu \). Then we may assume \( q' \ll h \), and for small \( q' \), the roots of (5.24) are approximately the (stable) wave modes

\[ \frac{\sigma}{-ik} \approx U \pm \left( \frac{g}{k} \tanh kh \right)^{1/2}, \]

(5.25)

and the erosive mode

\[ \sigma \approx -k^2Uq'[\sin k\delta + i \cos k\delta \tanh kh] \left[ \frac{F^2 - \coth kh}{kh} \right] \left[ \frac{F^2 - \tanh kh}{kh} \right], \]

(5.26)

where we define the Froude number by

\[ F = \frac{U}{\sqrt{gh}}. \]

(5.27)

For the erosive mode, the growth rate is

\[ \Re \sigma = -k^2Uq'\sin k\delta \tanh kh \left[ \frac{F^2 - \coth kh}{kh} \right] \left[ \frac{F^2 - \tanh kh}{kh} \right], \]

(5.28)

and the wave speed is

\[ -\frac{\Im \sigma}{k} = kUq' \cos k\delta \tanh kh \left[ \frac{F^2 - \coth kh}{kh} \right] \left[ \frac{F^2 - \tanh kh}{kh} \right]. \]

(5.29)
eigure 5.9: Instability diagram for the potential flow model. The regions marked
with a minus sign, above the upper curve and below the lower curve, are regions of
instability if $\delta < 0$, more specifically if $\sin k\delta < 0$. The marked distinction between
dunes and anti-dunes is based on the surface/bed phase relation (see (5.30)). Wave
motion is downstream if $\cos k\delta > 0$, upstream if $\cos k\delta < 0$.

This gives us the typical instability diagram shown in figure 5.9. For $\delta < 0$ (more
specifically, $\sin k\delta < 0$) the regions above and below the two curves are unstable,
corresponding to dunes and anti-dunes. The curves are given by $F^2 = \frac{\coth kh}{kh}$ and
$F^2 = \frac{\tanh kh}{kh}$, respectively.

The phase relation between surface and bed for the erosive bed is given by

$$\frac{\tilde{\zeta}}{s} \approx \frac{F^2 \text{sech} kh}{\left[ F^2 - \frac{\tanh kh}{kh} \right]}, \quad (5.30)$$

and this defines wave forms below the lower curve in figure 5.9 as dunes, and those
above as antidunes.

Figure 5.9 is promising, at least if $\sin k\delta < 0$, as it will predict both dunes and
anti-dunes. To get the wave speed positive, we need in fact to have $\cos k\delta > 0$, thus
$0 > k\delta > -\pi/2$ (we can take $-\pi < k\delta < \pi$ without loss of generality), whereas we
would generally want $k\delta < -\pi/2$ for anti-dunes to migrate backwards.
There is a serious problem with this model, beyond the fact that the phase shift \( \delta \) is arbitrarily included. The spatial delay is unlikely to provide a feasible model for nonlinear studies; indeed, we see that \( \text{Re} \sigma \sim k^2 \) at large \( k \), and in the unstable régime this is one of the hallmarks of ill-posedness.

Having said that, it will indeed turn out to be the case that a phase lead \( (\delta < 0) \) really is the cause of instability. A phase lead means that the stress, and thus the bedload transport, takes its maximum value on the upstream face of a bump in the bed. A phase lead will occur because of the effect of the bump on the turbulent velocity structure above, as we discuss further below. It can also occur through an effect of bedload inertia (see also question 5.7).

The choice of wave speed in this theory is unclear, since \( \cos k \delta \) can be positive or negative. The possibly more likely choice of a positive value implies positive wave speed.

### 5.4 St. Venant type models

Since river flow is typically modelled by the St. Venant equations, it is natural to try using such a model together with a bed erosion equation to examine the possibility of instability. This has the added advantage of being more naturally designed for fully nonlinear studies. A St. Venant/Exner model can be written in the form (cf. the footnote following (4.46))

\[
\begin{align*}
\partial_t s + q_x &= 0, \\
\partial_t h + (uh)_x &= 0, \\
\partial_t u + uu_x &= gS - \frac{fu^2}{h} - g\eta_x,
\end{align*}
\]

where \( S \) is the downstream slope, \( q = q(\tau), \tau = f \rho_w u^2, \) and \( \eta - s = h \). It is convenient to take advantage of the limit \( q \ll hu \), just as we did before, and we do so by first non-dimensionalising the equations. We choose scales as follows:

\[
s, x, h, \eta \sim h_0, \quad u \sim u_0, \quad q \sim q_0, \quad t \sim \frac{h_0^2}{q_0},
\]

and we choose \( h_0, u_0 \) by balancing terms as follows: \( uh \sim Q_0, gS \sim fu^2/h \); here \( Q_0 \) is the (prescribed) volume flow per unit width. We choose \( q_0 \) as the size of the bedload transport equation in (5.5).

With these scales, the dimensionless equations corresponding to (5.31) are

\[
\begin{align*}
\varepsilon \partial_t s + q_x &= 0, \\
\varepsilon \partial_t h + (uh)_x &= 0, \\
F^2(\varepsilon u_t + uu_x) &= -\eta_x + \delta \left( 1 - \frac{u^2}{h} \right), \\
h &= \eta - s,
\end{align*}
\]
where the parameters are

\[ F = \frac{u_0}{\sqrt{gh_0}}, \quad \varepsilon = \frac{q_0}{Q_0}, \quad \delta = S. \]

If we now suppose \( \varepsilon \ll 1 \) and \( \delta \ll 1 \), both of them realistic assumptions, then we have approximately

\[ uh = 1, \quad \frac{1}{2} F^2 u^2 + \eta = \frac{1}{2} F^2 + 1, \]

supposing that \( u, h \to 1 \) at large distances. Eliminating \( h \) and \( \eta \), we have

\[ s = 1 - \frac{1}{u} + \frac{1}{2} F^2 (1 - u^2), \]

whose form is shown in figure 5.10. In particular, \( s'(1) = (1 - F^2) \), so the basic state \( u = 1 \) corresponds to the left hand or right hand root of \( s(u) \) depending on whether the Froude number \( F < 1 \) or \( F > 1 \).

![Figure 5.10](image_url)  
**Figure 5.10:** \( s(u) \) as given by (5.36) for two typical cases of rapid and tranquil flow.

We also have

\[ \frac{ds}{d\eta} = \frac{F^2 - h^3}{F^2}, \]

so that small perturbations to \( h = 1 \) are out of phase (dunes) if \( F < 1 \) and in phase (anti-dunes) if \( F > 1 \). If we take the dimensionless bedload transport as \( q \approx \tau^{3/2} = u^3 \)
(the dimensionless basal stress having been scaled with \( f \rho u_0^2 \)), so that \( u = q^{1/3} \), then we see from (5.36) that \( s = s(q) \), and \( s(q) \) has the same shape as \( s(u) \), as shown in figure 5.10.

The whole model reduces to the single first order equation

\[
s'(q)q_t + q_x = 0. \tag{5.38}
\]

Disturbances to the uniform state \( q = 1 \) will propagate at speed \( v(q) = 1/s'(q) \), where \( v \) is shown in figure 5.11. For \( F < 1 \), \( v(1) > 0 \) and \( v'(1) > 0 \), thus waves in \( q \) (and thus \( s \)) propagate downstream and form forward facing shocks; this is nicely consistent with dunes. For \( F > 1 \), \( v < 0 \) and \( v'(1) \) is positive if \( F < 2 \), negative if \( F > 2 \) (see question 5.4). Backward facing shocks form, these are elevations in \( s \) if \( v' > 0 \).

![Figure 5.11: The wave speed \( v(q) = 3q^{4/3}/(1 - F^2 q) \) for the tranquil and rapid cases \( F = 0.5 \) and \( F = 1.5 \).](image)

Unfortunately, the hyperbolic equation does not admit instability. It is straightforward to insert a lag as before, by writing \( q(x, t) = q[s(x - \delta, t)] \), or equivalently \( s(x, t) = s[q(x + \delta, t)] \). Perturbation of

\[
s_t + q_x = 0, \quad q = q[s(x - \delta, t)], \tag{5.39}
\]

via

\[
s = \tilde{s} e^{i k x + \sigma t}, \quad q = 1 + \tilde{q} e^{i k x + \sigma t}, \tag{5.40}
\]

leads to

\[
\sigma \tilde{s} + i k \tilde{q} = 0,
\]

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\[ \bar{q} = q' e^{-ik\delta \bar{s}}, \]  

and thus
\[ \sigma = kq' [-\sin k\delta - i\cos k\delta]. \]

This requires \( \sin k\delta < 0 \) for instability if \( q'(s) > 0 \) \((F < 1)\) and \( \sin k\delta > 0 \) if \( q'(s) < 0 \) \((F > 1)\). The long wavelength limit of (5.26) in which \( kh \to 0 \) is precisely (5.42), bearing in mind that (5.26) is dimensional and that \( q' = dq/ds \) there, whereas \( q' = dq/du \) in (5.42).

### 5.5 A suspended sediment model

The shortcoming of both the potential model and the St. Venant/Exner model is the lack of a genuine instability mechanism. We now show that the inclusion of suspended load can produce instability. Ideally, we would hope to predict anti-dunes, since dunes certainly do not require suspended sediment transport. A St. Venant model including both bedload and suspended sediment transport is

\[
h_t + (uh)_x = 0, \\
\frac{\partial}{\partial t} (h c) + \frac{\partial}{\partial x} (h c u) = \rho_s (v_E - v_D), \\
(1 - n) \frac{\partial s}{\partial t} + \frac{\partial q_b}{\partial x} = -(v_E - v_D),
\]

where \( c \) is the column average concentration (mass per unit volume) of suspended sediment (written as \( \bar{c} \) earlier). The distinction between suspended sediment transport and bedload lies in the source terms due to erosion and deposition, \( v_E \) and \( v_D \), and it is these which will enable instability to occur. We have \( \eta - s = h \), and we suppose \( q_b = q_b(\tau) \), \( \tau = \int \rho_c u^2 \), whence \( q = q(u) \). Additionally (see (5.7) and (5.10)), we write

\[ v_E = v_s E, \quad \rho_s v_D = v_s c D, \]

and expect that \( E = E(u) \) and \( D = D(u) \), with \( E' > 0 \), \( D' < 0 \); typically \( E < 1 \), \( D > 1 \).

We scale (5.43) as before in (5.32), except that we choose the time scale \( t_0 \), downstream length scale \( x_0 \), and concentration scale \( c_0 \) via

\[ c_0 = \rho_s \frac{E_0}{D_0}, \quad t_0 = \frac{(1 - n)h_0}{v_s E_0}, \quad x_0 = \frac{Q_0}{v_s D_0}, \]

where we write

\[ E = E_0 E^*(u/u_0), \quad D = D_0 D^*(u/u_0), \]

where

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and choose \( E_0 \) and \( D_0 \) so that \( E^* \) and \( D^* \) are \( O(1) \), and so that these are consistent with typical observed suspended loads of \( 10 \text{ g l}^{-1} \). With this choice of scales, we obtain the dimensionless set of equations

\[
\begin{align*}
\eta - s &= h, \\
\varepsilon h_t + (uh)_x &= 0, \\
F^2(\varepsilon u_t + uu_x) &= \delta \left( 1 - \frac{u^2}{h} \right) - \eta_x, \\
h(\varepsilon c_t + uc_x) &= E^* - cD^*, \\
s_t + \beta q_x &= -(E^* - cD^*),
\end{align*}
\]

(5.47)

where the parameters \( \varepsilon, F, \delta \) and \( \beta \) are now given by

\[
\begin{align*}
\varepsilon &= \frac{E_0}{(1 - n)D_0} = \frac{c_0}{\rho_s(1 - n)}, & \delta &= \frac{u_0 S}{v_s D_0}, \\
F &= \frac{u_0}{(gh_0)^{1/2}}, & \beta &= \frac{q_0D_0}{Q_0E_0} = \frac{\rho_s q_0}{c_0 Q_0}.
\end{align*}
\]

(5.48)

Here \( q_{b0} \) is the scale for \( q_b \) rather than \( q = q_b/(1 - n) \). The Froude number is the same as before, but the parameters \( \varepsilon \) and \( \delta \) are different: \( \varepsilon \) is a measure of the suspended sediment density relative to the bed density, and is always small; \( \delta \) is the ratio of the (small) bed slope to the ratio of settling velocity to stream velocity. For more rapidly flowing streams, we might expect \( \delta \sim 1 \). However, if we suppose that wavelengths of anti-dunes are comparable to the depth (so \( x_0 \sim h_0 \), then (5.45) implies \( \delta \sim S \ll 1 \). Thus \( \delta \sim 1 \) implies \( x_0 \sim h_0/S \gg h_0 \). The parameter \( \beta \) is a direct measure of the ratio of bedload \( (\rho_s q_0) \) to suspended load \( (c_0 Q_0) \). For \( \beta \gg 1 \), we would revert to our preceding bedload model and its scaling, and neglect the suspended load. If we adopt the Meyer-Peter/Müller relation in (5.5) and (5.6), then (noting that \( fu_0^2 = gSh_0 \))

\[
q_{b0} = \frac{K \rho_l}{\Delta \rho} (ghS)^{3/2},
\]

(5.49)

and we can write

\[
\beta = \left\{ \frac{K \rho_l}{(1 - n)\Delta \rho} \right\} \frac{(ghS)^{3/2}}{\varepsilon F};
\]

(5.50)

both small or large values are possible.

To analyse (5.47), we ignore bedload (put \( \beta = 0 \)) and take \( \varepsilon \to 0 \). Then

\[
\eta = h + s, \quad uh = 1,
\]

(5.51)

so that

\[
\begin{align*}
c_x &= E^*(u) - cD^*(u) = -s_t, \\
\frac{\partial}{\partial x} \left[ \frac{1}{2} F^2 u^2 + \frac{1}{u} + s \right] &= \delta (1 - u^3).
\end{align*}
\]

(5.52)
If, in addition, \( \delta \ll 1 \), then, taking \( s = 0 \) when \( h = 1 \),

\[
s = s(u) = \frac{1}{2} F^2 (1 - u^2) + 1 - \frac{1}{u},
\]

and the entire suspended load model is

\[
s'(u) \frac{\partial u}{\partial t} = c D^*(u) - E^*(u) = - \frac{\partial c}{\partial x}.
\]

The function \( s(u) \) is the same as we derived before in (5.36) and shown in figure 5.10. We can in fact write (5.54) as a single equation for \( u \), by eliminating \( c \); this gives

\[
c = \frac{E^*(u)}{D^*(u)} + \frac{s'(u)}{D^*(u)} \frac{\partial u}{\partial t},
\]

\[
s'(u) \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left[ \frac{E^*(u)}{D^*(u)} + \frac{s'(u)}{D^*(u)} \frac{\partial u}{\partial t} \right] = 0,
\]

and the equation for \( u \) (or the pair for \( u, c \)) is of hyperbolic type. Note that natural initial-boundary conditions for (5.54) are to prescribe \( u \) at \( t = 0, x > 0 \), and \( c \) at \( x = 0, t > 0 \).

Let us examine the stability of the steady state \( u = 1, c = 1 \). We put

\[
u = 1 + \text{Re} \left( U e^{ikx + \sigma t} \right), \quad c = 1 + \text{Re} \left( C e^{ikx + \sigma t} \right),
\]

and linearise, to obtain (noting \( E^*(1) = D^*(1) = 1 \))

\[
ikiC = [E^*(1) - D^*(1)]U - C = -\sigma s'(1),
\]

and thus

\[
\sigma = \left[ \frac{E^*(1) - D^*(1)}{s'(1)} \right] \left( \frac{-k^2 - ik}{1 + k^2} \right).
\]

If we suppose \( E^{*'} > 0, D^{*'} < 0 \) as previously suggested, then this model implies instability (\( \text{Re} \sigma > 0 \)) for \( s'(1) < 0 \), i.e. \( F > 1 \), and that the wave speed is \(-\text{Im} (\sigma)/k < 0\); thus this theory predicts upstream-migrating anti-dunes.

Two features suggest that the model is not well posed if \( F > 1 \). The first is the instability of arbitrarily small wavelength perturbations; the second is that the unstable waves propagate upstream, although the natural boundary condition for \( c \) is prescribed at \( x = 0 \).

Numerical solutions of (5.54) are consistent with these observations. In solving the nonlinear model (5.54) in \( 0 < x < \infty \), we note that

\[
\frac{d}{dt} \int_0^\infty s(u) \, dx = -[c]_0^\infty,
\]

which simply represents the net erosion of the bed downwards if the sediment flux at infinity is greater than at zero. It thus makes sense to fix the initial boundary conditions so that

\[
c = 1 \quad \text{on} \quad x = 0,
\]

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\[ u \to 1 \text{ as } x \to \infty, \ t = 0. \] (5.60)

For \( F < 1 \), numerical solutions are smooth and approach the stable solution \( u = c = 1 \). However, the solutions are numerically unstable for \( F > 1 \), and \( u \) rapidly blows up, causing breakdown of the solution.

Some further insight into this is gained by consideration of the solution at \( x = 0 \). If \( c = c_0(t) \) on \( x = 0 \) and \( u = u_0(x) \) on \( t = 0 \), then we can obtain \( u \) on \( x = 0 \) from (5.55), by solving the ordinary differential equation

\[ \frac{\partial u}{\partial t} + \frac{E^*(u)}{s'(u)} = \frac{D^*(u)}{s'(u)} c_0(t) \] (5.61)

with \( u = u_0(0) \) at \( t = 0 \). If we suppose that \( c = 1 \) at \( x = 0 \), then it is easy to show that if \( F < 1 \) and \( u(0, 0) < 1/F^{2/3} \), then \( u(0, t) \to 1 \) as \( t \to \infty \). If on the other hand, \( F > 1 \) and \( u(0, 0) < 1 \), then \( u(0, t) \to 1/F^{2/3} \) in finite time, and the solution breaks down as \( \partial u/\partial t \to \infty \); if \( u(0, 0) > 1 \), then \( u(0, t) \to \infty \), again in finite time if, for example, \( E^* \propto u^3 \). More generally, breakdown of the solution when \( F > 1 \) occurs in one of these ways at some positive value of \( x \). Thus this suspended sediment model shares the same weakness of the phase shift model in not appearing to provide a well posed nonlinear model.

### 5.6 Eddy viscosity model

The relative failure of the models above to explain dune and anti-dune formation led to the consideration of a full fluid flow model, in which, rather than supposing that the flow is shear free and that viscous effects were confined to a turbulent boundary layer, rotational effects were considered, and a model of turbulent shear flow incorporating an eddy viscosity, together with the Exner equation for bedload transport, was adopted. This allows for a linear stability analysis of the uniform flow over a flat bed via the solution of a suitable Orr-Sommerfeld equation. We shall in fact proceed in somewhat more generality. As an observation, fully-formed dunes have relatively small height to length ratios, and thus the fluid flow over them can be approximately linearised. Although we use a linear approximation to derive the stress at the bed, we may retain the nonlinear Exner equation for example. In this way we may derive a nonlinear evolution equation for bed elevation.

#### 5.6.1 Orr-Sommerfeld equation

Suppose, therefore, that we have two-dimensional turbulent flow down a slope of gradient \( S \), governed by the Reynolds equations

\[ u_t + uu_x + uw_z = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u + gS, \]

\[ w_t + uw_x + ww_z = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 w - g(1 - S^2)^{1/2}, \]
where \((u, w)\) are the velocity components and \(\nu_T\) is an eddy viscosity associated with the Reynolds stress terms, such as prescribed in (B.9). In the second equation, we can take \(g(1 - S^2)^{1/2} \approx g\) since \(S\) is small.

We consider perturbations to a basic shear flow \(u(z)\) in \(s < z < \eta\) which satisfies (5.62) with \(\nu_T\) taken as constant. (Later, we will study a more realistic eddy viscosity model.) It is convenient first of all to non-dimensionalise the equations (5.62). In the basic uniform state, with \(s = 0\) and \(\eta = \bar{h}\), the shear flow satisfies

\[
u_T \frac{\partial u}{\partial z} = g\bar{h} - z, \tag{5.63}\]

whence

\[
u_T \frac{\partial u}{\partial z} = \frac{gS}{\nu_T} \left(\bar{h} - \frac{1}{2}z^2\right), \tag{5.64}\]

and the column mean flow is

\[ar{u} = \frac{1}{h} \int_0^h u \, dz = \frac{gS}{3\nu_T} \bar{h}^2. \tag{5.65}\]

Taking \(\nu_T = \varepsilon_T \bar{u} \bar{h}\), we find that the basal shear stress is

\[	au = \rho_w \nu_T \left. \frac{\partial u}{\partial z} \right|_0 = f \rho_w \bar{u}^2, \tag{5.66}\]

where \(f = 3\varepsilon_T\). This gives the relationship between the empirical \(f\) and the semi-analytic \(\varepsilon_T\). If the bed and hence the flow is perturbed, we would only retain constant \(\nu_T\) if the volume flux per unit width is the same; this we therefore assume.

We now non-dimensionalise the variables by writing

\[(u, w) \sim \bar{u}, \quad (x, z) \sim \bar{h}, \quad t \sim \bar{h}/\bar{u}, \quad p - pg(\bar{h} - z) \sim \rho_w \bar{u}^2. \tag{5.67}\]

The dimensionless equations are

\[
u_T \frac{\partial u}{\partial z} = g\bar{h} - z, \tag{5.63}\]

\[
u_T \frac{\partial u}{\partial z} = \frac{gS}{\nu_T} \left(\bar{h} - \frac{1}{2}z^2\right), \tag{5.64}\]

\[
u_T \frac{\partial w}{\partial z} = g\bar{h} - z, \tag{5.63}\]

\[
u_T \frac{\partial w}{\partial z} = \frac{gS}{\nu_T} \left(\bar{h} - \frac{1}{2}z^2\right), \tag{5.64}\]

and the parameters are a turbulent Reynolds number and the Froude number:

\[R = \frac{\bar{h}}{\nu_T}, \quad F = \frac{\bar{u}}{\sqrt{g\bar{h}}} \tag{5.69}\]

The dimensionless basic velocity profile is then

\[u = \frac{gS\bar{h}^2}{\nu_T \bar{u}} \left(\bar{h} - \frac{1}{2}z^2\right), \tag{5.70}\]
and the dimensionless mean velocity is, by definition of $\bar{u}$,

$$1 = \frac{gS\bar{h}^2}{3\nu_T\bar{u}}. \quad (5.71)$$

Since

$$\nu_T = \varepsilon_T\bar{u}\bar{h} = \frac{1}{3}f\bar{u}\bar{h}, \quad (5.72)$$

this requires

$$\bar{u} = \left(\frac{gS\bar{h}}{f}\right)^{1/2}. \quad (5.73)$$

In particular, the dimensionless basic velocity profile is

$$u = U(z) = 3\left(z - \frac{1}{2}z^2\right). \quad (5.74)$$

We now suppose that $s$ and $\eta$ are perturbed by small amounts; we may thus linearise (5.68). We put

$$(u, w) = (U(z) + \psi_z, -\psi_x), \quad (5.75)$$

whence it follows for small $\psi$ that $\psi$ satisfies the steady state Orr-Sommerfeld equation

$$U\nabla^2\psi_x - U''\psi_x = R^{-1}\nabla^4\psi, \quad (5.76)$$

where we assume stationary solutions in view of the anticipated fact that $s$ evolves on a slower time scale.

The condition of zero pressure at $z = \eta$ is linearised to be

$$\eta = 1 + \frac{F^2p}{z=1}. \quad (5.77)$$

If $F^2$ is small, then we may take $\eta$ to be constant, and we do so as we are primarily interested in dunes. However, the dimensionless pressure $p$ is only determined up to addition of an arbitrary constant, which implies that the value of the constant $\eta$ is unconstrained. This represents the vertical translation invariance of the system. If a uniform perturbation to $s$ is made, then the response of the (uniform) stream is to raise the surface by the same amount. We can remove the ambiguity by prescribing $\eta = 1$, with the implication that the mean value of $s$ is required to be zero.

The other boundary conditions on $z = s$ and $z = 1$ are no slip at the base, no shear stress at the top, and the perturbed volume flux is zero. These imply

$$\psi = 0, \ \psi_{zz} = 0 \ \text{on} \ z = 1,$$

$$\int_0^s U(z) \, dz + \psi = 0, \ U + \psi_z = 0 \ \text{on} \ z = s. \quad (5.78)$$

Linearisation of this second pair about $z = 0$ gives

$$\psi = 0, \ \psi_z = -U_0's \ \text{on} \ z = 0, \quad (5.79)$$
where $U'_0 = U'(0)$. Our aim is now to solve (5.76) with (5.78) and (5.79) to calculate the perturbed shear stress. The dimensional basal shear stress is then

\[
\tau = \rho \varepsilon_T u_0^2 U'_0 \left[ 1 + s \frac{U''_0}{U'_0} + \frac{1}{U'_0} \psi_{zz} \right],
\]

(5.80)

and since $f = 3 \varepsilon_T = \varepsilon_T U'_0$, we may write this as

\[
\tau = f \rho \bar{u}^2 \left[ 1 + s \frac{U''_0}{U'_0} + \frac{1}{U'_0} \psi_{zz} \right].
\]

(5.81)

The problem to solve for $\psi$ is linear and inhomogeneous, and so we suppose that

\[
s = \int_{-\infty}^{\infty} \hat{s}(k) e^{ikx} \, dk, \quad \psi = \int_{-\infty}^{\infty} \hat{\psi}(k) e^{ikx} \, dk.
\]

(5.82)

(Note $\hat{s}$ will evolve slowly in time.) For each wave number $k$, we obtain

\[
\hat{k} \left[ U(\psi'' - k^2 \hat{\psi}) - U'' \hat{\psi} \right] = \frac{1}{R} \left[ \psi'^v - 2k^2 \psi'' + k^4 \hat{\psi} \right],
\]

(5.83)

with boundary conditions

\[
\hat{\psi} = \hat{\psi}'' = 0 \quad \text{on} \quad z = 1,
\]

\[
\hat{\psi} = 0, \quad \hat{\psi}' = -U'_0 \hat{s} \quad \text{on} \quad z = 0,
\]

(5.84)

and thus we finally define

\[
\hat{\psi} = -U'_0 \hat{s} \Psi(z, k),
\]

(5.85)

where $\Psi$ satisfies the canonical problem

\[
\hat{k} \left[ U(\Psi'' - k^2 \Psi) - U'' \Psi \right] = \frac{1}{R} \left[ \Psi'^v - 2k^2 \Psi'' + k^4 \Psi \right],
\]

\[
\Psi = \Psi'' = 0 \quad \text{on} \quad z = 1,
\]

\[
\Psi = 0, \quad \Psi' = 1 \quad \text{on} \quad z = 0.
\]

(5.86)

In terms of $\Psi$, the basal (dimensional) shear stress is

\[
\tau = f \rho \bar{u}^2 \left[ 1 - s - \int_{-\infty}^{\infty} e^{ikx} \hat{s}(k) \Psi''(0, k) \, dk \right].
\]

(5.87)

Using the convolution theorem, this is

\[
\tau = f \rho \bar{u}^2 \left[ 1 - s + \int_{-\infty}^{\infty} K(x - \xi) s'(\xi) \, d\xi \right],
\]

(5.88)

where $s' = \partial s/\partial x$, and

\[
K(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Psi''(0, k)}{ik} e^{ikx} \, dk.
\]

(5.89)
Depending on $K$, we can see how $\tau$ may depend on displaced values of $s$. The form of (5.88) illustrates our previous discussion of the vertical translation invariance of the system. For a possible uniform perturbation $s = \text{constant}$, we would obtain a modification to the basic friction law, $\tau = f \rho \bar{u}^2$. This is excluded by enforcing the condition that $s$ has zero mean in $x$, 

$$\lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} s(x) \, dx = 0, \tag{5.90}$$

which corresponds (for a periodic bed) to prescribing

$$\tilde{s}(0) = 0. \tag{5.91}$$

To determine $K$, we need to know the solution of (5.86) for all $k$. In general, the problem requires numerical solution. However, note that $R = 1/\varepsilon_T$, and is reasonably large (for a value $f = 0.005$, $R = 3/f = 600$). This suggests that a useful means of solving (5.86) may be asymptotically, in the limit of large $R$. The fact that we can obtain analytic expressions for $\Psi''(0, k)$ means this is useful even when $R$ is not dramatically large, as here.

The solution of the Orr-Sommerfeld equation at large $R$ has a long pedigree, and it is a complicated but mathematically interesting problem. We devote Appendix C to finding the solution. We find there that, for $k > 0$,

$$\Psi''(0, k) \approx -3(i k RU_0')^{1/3} \text{Ai}(0) + O(1), \tag{5.92}$$

where $\text{Ai}$ is the Airy function. For $k < 0$, $\Psi''(0, k) = \overline{\Psi''(0, -k)}$, and this leads to

$$\frac{\Psi''(0, k)}{ik} \approx \begin{cases} -ce^{-\pi/3} |k|^{-2/3}, & k > 0 \\ -ce^{\pi/3} |k|^{-2/3}, & k < 0, \end{cases} \tag{5.93}$$

where

$$c = 3(RU_0')^{1/3} \text{Ai}(0), \tag{5.94}$$

and $c \approx 1.54R^{1/3}$ for $U_0' = 3$, as $\text{Ai}(0) = \frac{1}{3^{2/3} \Gamma(\frac{2}{3})} \approx 0.355$. From (5.89), we find

$$K(x) = \frac{c}{\pi} \int_0^\infty \frac{\cos[kx - \frac{\pi}{3}] \, dk}{k^{2/3}}. \tag{5.95}$$

Evaluating the integral,$^3$ we obtain the simple formula

$$K(x) = \begin{cases} \frac{\mu}{x^{1/3}}, & x > 0, \\ 0, & x < 0, \end{cases} \tag{5.96}$$

$^3$How do we do that? The blunt approach is to consult Gradshteyn and Ryzhik (1980), where the relevant formulae are on page 420 and 421 (items 4 and 9 of section 3.761). The quicker way, using complex analysis, is to evaluate $\int_0^\infty \theta^{\nu-1} e^{\nu} \, d\theta$ (after a simple rescaling of $k$, $k|x| = \theta$) by rotating the contour by $\pi/2$ and using Jordan’s lemma. Thus $\int_0^\infty \theta^{\nu-1} e^{\nu} \, d\theta = \Gamma(\nu)e^{\nu\pi/2}$.  

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where
\[ \mu = \frac{3^{2/3} R^{1/3}}{\{\Gamma(\frac{2}{3})\}^2} \approx 1.13 R^{1/3}. \]  

For stability purposes, note that
\[ K = \int_{-\infty}^{\infty} \hat{K}(k) e^{ikx} \, dk; \]  
\[ \hat{K} = -\frac{\Psi''(0,k)}{2\pi ik} = \frac{c \exp \left[ -\frac{i\pi}{3} \text{sgn} \, k \right]}{2\pi |k|^{2/3}}. \]  

5.6.2 Orr-Sommerfeld-Exner model

We now reconsider (5.33), which we can write in the form
\[ s_t + q_x = 0, \]
\[ \varepsilon h_t + (uh)_x = 0, \]
\[ F^2(\varepsilon u_t + uu_x) = -\eta_x + \delta \left( 1 - \frac{\tau}{h} \right), \]
\[ h = \eta - s. \]  

Here \( \tau \) is the local basal stress, scaled with \( f \rho_w u_0^2 \). We suppose \( q = q(\tau) \), so that the Exner equation is
\[ \frac{\partial s}{\partial t} + q'(\tau) \frac{\partial s}{\partial x} = 0. \]  

It is tempting to suppose that, writing \( \bar{u} = u_0u \),
\[ \tau = u^2 \left[ 1 - s + \int_{-\infty}^{\infty} K(x - \xi) \frac{\partial s}{\partial \xi}(\xi, t) \, d\xi \right]. \]  

We would then have, with \( \varepsilon \ll 1 \) and \( \eta = 1, u \approx \frac{1}{1 - s} \) in (5.102). There is a subtle point here concerning the modified stress. Insofar as we may wish to describe different atmospheric or fluvial conditions (e.g., the difference between strong and weak winds at different times of day, or rivers in normal or flood stage), we do want to allow different choices of \( \bar{u} \). However, such conditions also imply different values of \( \bar{h} \), and the basis of the solution for the perturbed stress is that the mean depth (and thus the mean velocity) do not vary. The value of \( u = \frac{1}{1 - s} \) is a local column average, whereas the \( u \) in (5.102) is in addition a horizontal average. Thus, given \( u_0 = \bar{u} \) and \( h_0 = \bar{h} \), we define
\[ \tau \approx 1 - s + \int_{-\infty}^{\infty} K(x - \xi) \frac{\partial s}{\partial \xi}(\xi, t) \, d\xi, \]  

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and the model consists of the Exner equation (5.101) and the Orr-Sommerfeld stress formula (5.103). Variable $\bar{u}$ is simply manifested in differing time scales for the Exner equation.

We linearise by writing $\tau = 1 + T$, and then

$$s = \int_{-\infty}^{\infty} \hat{s}(k, t)e^{ikx} dk, \quad T = \int_{-\infty}^{\infty} \hat{T}(k, t)e^{ikx} dk,$$

so that

$$\hat{s}_t + ikq'(1)\hat{T} = 0,$$

$$\hat{T} = -\hat{s} + 2\pi \hat{K} ik\hat{s},$$

and thus, using (5.99), solutions are proportional to $e^{\sigma t}$, where

$$\sigma = q'(1)[2\pi k^2 \hat{K} + ik].$$

When $\text{Re} \hat{K} > 0$, as for (5.99), the steady state is unstable, with $\text{Re} \sigma \sim k^{4/3}$ as $k \to \infty$. Specifically, the growth rate is

$$\text{Re} \sigma = \frac{1}{2} q'(1)c|k|^{4/3},$$

while the wave speed is

$$-\frac{\text{Im} \sigma}{k} = q'(1) \left( \frac{1}{2} \sqrt{3} c|k|^{1/3} - 1 \right);$$

thus waves move downstream (except for very long waves).

### 5.6.3 Well-posedness

The effect of (5.103) is to cause increased $\tau$ where $s_x$ is positive, on the upstream slopes of bumps. Since $u$ is in phase with $s$, this implies $\tau$ leads $u$ (i.e., $\tau$ is a maximum before $s$ is); it is this phase lead which causes instability. However, the unbounded growth rate at large wave numbers is a sign of ill-posedness. Without some stabilising mechanism, arbitrarily small disturbances can grow arbitrarily rapidly. In reality, another effect of bed slope is important, and that is the fact that sediment wants to roll downslope: in describing the Meyer-Peter/Müller result, no attention was paid to the variations of bed slope itself.

For a particle of diameter $D_s$ at the bed, the streamflow exerts a force of approximately $\tau D_s^2$ on it, and it is this force which causes motion. On a slope, there is an additional force due to gravity, approximately $-\Delta \rho g D_s^2 s_x$. Thus the net stress causing motion is actually

$$\tau - \Delta \rho g D_s s_x.$$

In dimensionless terms, we therefore modify the bedload transport formula by writing

$$q = q(\tau_e), \quad \tau_e = \tau - \beta s_x,$$
Figure 5.12: Development of the dune instability from an initial perturbation near
\( x = 0 \) obtained by solving (5.116) using \( q(\tau) = \tau^{3/2} \), \( K = \frac{\mu}{x^{1/3}} \) when \( x > 0 \), \( K = 0 \) otherwise, with parameters \( \mu = 9.57 \) and \( D = 4.3 \). Separation occurs in this figure when \( t = 0.8 \), after which the computation is continued as described in the notes at
the end of the chapter. Figure kindly provided by Mark McGuinness.

where

\[
\beta = \frac{\Delta \rho D_s}{\rho_w h_0 S}.
\] (5.111)

Typical values in water are \( \Delta \rho/\rho_w \approx 2 \), \( D_s \sim 10^{-3} \) m, \( h_0 \sim 2 \) m, \( S \sim 10^{-3} \), whence \( \beta \approx 4 \); generally we will suppose that \( \beta \sim O(1) \).

The effect of this is to replace the definition of \( \tau \) in (5.103) by

\[
\tau_c = 1 - s + \int_{-\infty}^{\infty} K(x - \xi) \frac{\partial s}{\partial \xi}(\xi, t) d\xi - \beta s_x,
\] (5.112)

(together with (5.101)) and in the stability analysis, \( \tilde{T} = \tilde{s}[-1 + 2\pi i k \tilde{K} - ik^2] \), whence

\[
\sigma = q'(1) \left[ -ik \left\{ \frac{1}{2} \sqrt{3} c |k|^{1/3} - 1 \right\} + \frac{1}{2} c |k|^{4/3} - \beta k^2 \right].
\] (5.113)

This exhibits the classical behaviour of a well-posed model. The system is stable at
high wavenumber, and the maximum growth rate is at \( k = \left( \frac{c}{3\beta} \right)^{3/2} \). This would be
the expected preferred wavenumber of the instability.

Figure 5.12 shows a numerical solution of the nonlinear Exner equation, showing
the growth of dunes from an initially localised disturbance. Because the expression
in (5.112) is only valid for small \( s \), we can equivalently write

\[
q(\tau_c) = q(\tau - \beta s_x) \approx q(\tau) - Ds_x,
\] (5.114)
where
\[
D = \beta q'(\tau) \approx \beta q'(1),
\]  
(5.115)
and the equation has been solved in this form, with the diffusion coefficient \(D\) taken
as constant, i.e., \(s\) satisfies the equation
\[
\frac{\partial s}{\partial t} + \frac{\partial}{\partial x} q \left[ 1 - s + K \ast s_x \right] = D \frac{\partial^2 s}{\partial x^2}.
\]  
(5.116)

As the dunes grow, the model becomes invalid when \(\tau - 1 \sim O(1)\), and this happens
when \(s \sim \frac{1}{\mu}\). This is a representative value for the elevation of both fluvial and aeolian
dunes, and is suggestive of the idea that it is the approach of \(\tau\) towards zero which
controls eventual dune height. Additionally, when \(\tau\) reaches zero, separation occurs,
and the model becomes invalid. Possible ways for dealing with this are outlined in
the notes at the end of the chapter. A further issue is that the derivation of (5.112)
becomes invalid when \(s \sim \frac{1}{\mu}\), because then the thickness of the viscous boundary
layer in the Orr–Sommerfeld equation becomes comparable to the elevation of the
dunes. This implies that the Orr–Sommerfeld equation should now be solved in a
domain where the lower boundary can not be linearised about \(z = 0\), and the Fourier
method of solution can no longer be implemented. It is not clear whether this will
fundamentally change the nature of the resultant formula for the stress.

The numerical method used to solve (5.116) is a spectral method. Spectral meth-
oods for evolution equations of this sort are convenient, particularly when the integral
term is of convolution type, but they confuse the issue of what appropriate boundary
conditions for such equations should be. In the present case, it is not clear. For ae-
olian dunes, it is natural to pose conditions at a boundary representing a shore-line,
but it is then less clear how to deal with the integral term. The derivation of this term
already presumes an infinite sand domain, and it seems this is one of those questions
akin to the issue of posing boundary conditions for averaged equations, for example
for two-phase flow, where a hidden interchange of limits is occurring.

5.7 Mixing-length model for aeolian dunes

Measurements of turbulent fluid flow in pipes, as well as air flow in the atmosphere
(and also in wind tunnels), show that the assumption of constant eddy viscosity is
not a good one, and the basic shear velocity profile is not as simple as assumed in the
preceding section. In actual fact, the concept of eddy viscosity introduced by Prandtl
was based on the idea of momentum transport by eddies of different sizes, with the
transport rate (eddy viscosity) being proportional to eddy size. Evidently, this must
go to zero at a solid boundary, and the simplest description of this is Prandtl’s mixing
length theory, described in appendix B. In this section, we generalise the previous
approach a little to allow for such a spatially varying eddy viscosity, and we specifically
consider the case of aeolian dunes, in which a kilometre deep turbulent boundary layer
flow is driven by an atmospheric shear flow.
5.7.1 Mixing-length theory

The various forms of sand dunes in deserts were discussed earlier; the variety of shapes can be ascribed to varying wind directions, a feature generally absent in rivers. Another difference from the modelling point of view is that the fluid atmosphere is about ten kilometres in depth, and the flow in this is essentially unaffected by the underlying surface, except in the atmospheric boundary layer, of depth about a kilometre, wherein most of the turbulent mixing takes place. Within this boundary layer, there is a region adjoining the surface in which the velocity profile is approximately logarithmic, and this region spans a range of height from about forty metres above the surface to the ‘roughness height’ of just a few centimetres or millimetres above the surface.

Consider the case of a uni-directional mean shear flow $u(z)$ past a rough surface $z = 0$, where $z$ measures distance away from the surface. If the shear stress is constant, equal to $\tau$, then we define the friction velocity $u_*$ by

$$u_* = (\tau/\rho)^{1/2},$$

(5.117)

where $\rho$ is density. Observations support the existence near the surface of a logarithmic velocity profile of the form

$$u = \frac{u_*}{\kappa} \ln \left( \frac{z}{z_0} \right),$$

(5.118)

where the Von Kármán constant $\kappa \approx 0.4$, and $z_0$ is known as the roughness length: it represents the effect of surface roughness in bringing the average velocity to zero at some small height above the actual surface. Since $z_0$ is a measure of actual roughness, a typical value for a sandy surface might be $z_0 = 10^{-3}$ m.

Prandtl’s mixing length theory provides a motivation for (5.118). If we suppose the motion can be represented by an eddy viscosity $\eta$, so that

$$\tau = \eta \frac{\partial u}{\partial z},$$

(5.119)

then Prandtl proposed

$$\eta = \rho l^2 \left| \frac{\partial u}{\partial z} \right|, \quad l = \kappa z,$$

(5.120)

from which, indeed, (5.118) follows. The quantity $l = \kappa z$ is called the mixing length. Prandtl’s theory works well in explaining the logarithmic layer, and in extension it explains pipe flow characteristics very well; but it has certain drawbacks. The two obvious ones are that it is not frame-invariant; however, this would be easily rectified by replacing $|\partial u/\partial z|$ by the second invariant $2\dot{\varepsilon}$, where $2\dot{\varepsilon}^2 = \dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}$, and $\dot{\varepsilon}_{ij}$ is the strain rate tensor. Also not satisfactory is the rather loosely defined mixing length, which becomes less appropriate far from the boundary, or in a closed container.

\[4\] A better recipe would be $u = \frac{u_*}{\kappa} \ln \left( \frac{z + z_0}{z_0} \right)$, which allows no slip at $z = 0$. See also question 5.11.
Despite such misgivings, we will use a version of the mixing length theory to see how it deviates from the constant eddy viscosity assumption.

We want to see how to solve a shear flow problem in dimensionless form. To this end, suppose for the moment that we fix \( u = U_\infty \) on \( z = d \). Then \( U_\infty = (u^*/\kappa) \ln(d/z_0) \) determines \( u* \) (and thus \( \tau \)), and we can define a parameter \( \varepsilon \) by

\[
\varepsilon = \frac{u_*}{U_\infty} = \frac{\kappa}{\ln(d/z_0)}.
\]

(5.121)

For \( d = 10^3 \) m, \( z_0 = 10^{-3} \) m, \( \kappa = 0.4 \), \( \varepsilon \approx 0.03 \). Writing \( u \) in terms of \( U_\infty \) rather than \( u* \) yields

\[
u = U_\infty \left[ 1 + \frac{\varepsilon}{\kappa} \ln \left( \frac{z}{d} \right) \right].
\]

(5.122)

Note also that the basic eddy viscosity is then

\[
\eta = \varepsilon \rho U_\infty d \left( \frac{\kappa z}{d} \right),
\]

(5.123)

and the shear stress is

\[
\tau = \varepsilon^2 \rho U_\infty^2.
\]

(5.124)

We shall use these observations in scaling the equations. For the atmospheric boundary layer, it seems appropriate to assume that \( U_\infty \) is prescribed from the large scale model of atmospheric flow (cf. chapter 3), and that \( d \) is the depth of the planetary boundary layer. Of course, this may be an oversimplified description.

### 5.7.2 Turbulent flow model

Again we assume a mean two-dimensional flow \((u, 0, w)\) with horizontal coordinate \( x \) and vertical coordinate \( z \) over a surface topography given by \( z = s \). The basic equations are

\[
\begin{align*}
    u_x + w_z & = 0, \\
    \rho(uu_x + uu_z) & = -p_x + \tau_{1x} + \tau_{3z}, \\
    \rho(uw_x + uw_z) & = -p_z + \tau_{3x} - \tau_{1z},
\end{align*}
\]

(5.125)

where \( \tau_1 = \tau_{11} \) and \( \tau_3 = \tau_{13} \) are the deviatoric Reynolds stresses, and are defined, we suppose, by

\[
\begin{align*}
    \tau_1 & = 2\eta u_x, \\
    \tau_3 & = \eta (u_z + w_x).
\end{align*}
\]

(5.126)

We ignore gravity here, so that the pressure is really the deviation from the hydrostatic pressure. Our choice of the eddy viscosity \( \eta \) will be motivated by the Prandtl mixing length theory (5.120), but we postpone a precise specification for the moment.

\[\text{Note that this definition of } \varepsilon \text{ is unrelated to its previous definition and use, as for example in (5.100).}\]
The basic flow then dictates how we should non-dimensionalise the variables. We do so by writing
\[ u = U_\infty (1 + \varepsilon u^*), \quad w \sim \varepsilon U_\infty, \quad x, z \sim d, \]
\[ \tau_1, \tau_3 \sim \varepsilon^2 \rho U_\infty^2, \quad \eta \sim \varepsilon \rho d U_\infty, \quad p \sim \varepsilon \rho U_\infty^2, \] (5.127)
and then the dimensionless equations are (dropping the asterisk on \( u^* \))
\[ u_x + w_z = 0, \]
\[ u_x + p_x = \varepsilon [\tau_1 x + \tau_3 x - \{ w u_x + w w_x \}], \]
\[ w_x + p_z = \varepsilon [\tau_3 x - \tau_1 x - \{ w w x + w w_z \}], \]
\[ \tau_3 = \eta (u_z + w_z), \]
\[ \tau_1 = 2 \eta u_x. \] (5.128)

### 5.7.3 Boundary conditions

The depth scale of the flow \( d \) is, we suppose, the depth of the atmospheric boundary layer, of the order of hundreds of metres to a kilometre. Above the boundary layer, there is an atmospheric shear flow, and we suppose that \( u \sim u_0(z), \ w \to 0, \ p \to 0 \) as \( z \to \infty \). The choice of \( u_0 \) is determined for us by the choice of \( \eta \), as is most easily seen from the case of a uniform flow where \( \partial u / \partial z = \tau / \eta \). The correct boundary condition to pose at large \( z \) is to prescribe the shear stress delivered by the main atmospheric flow, and this can be taken to be \( \tau_3 = 1 \) by our choice of stress scale. Thus we prescribe
\[ \tau_3 \to 1, \ w \to 0, \ p \to 0 \text{ as } z \to \infty. \] (5.129)

Next we need to prescribe conditions at the surface. This involves two further length scales, the length \( L \) and amplitude \( H \) of the surface topography. Since we observe dunes often to have lengths in the range 100–1000 m, and heights in the range 2–100 m, we can see that there are two obvious distinguished limits, \( L = d, \ H = \varepsilon d \), and it is most natural to use these in scaling the surface \( s \). In fact since dunes are self-evolving it seems most likely that they will select length scales already present in the system. Thus, we suppose that in dimensionless terms the surface is \( z = \varepsilon s(x) \), and longer, shorter, taller or smaller dunes can always be introduced as necessary later, by rescaling \( s \). The surface boundary conditions are then taken to be (recalling the definition of the roughness length)
\[ u = -\frac{1}{\varepsilon}, \ w = 0 \text{ on } z = \varepsilon s + z_0^*, \] (5.130)
where
\[ z_0^* = \frac{z_0}{d} = e^{-\kappa / \varepsilon}. \] (5.131)

For completeness, we need to specify horizontal boundary conditions, for example at \( x = \pm \infty \). We keep these fairly vague, beyond requiring that the variables remain bounded. In particular, we do not allow unbounded growth of velocity or pressure.

---

6The modelling alternative is to specify velocity conditions on a lid at \( z = 1 \).
5.7.4 Eddy viscosity

Prandtl’s mixing length theory in scaled units would imply

\[ \eta = \kappa^2 (z - \varepsilon s)^2 \frac{\partial u}{\partial z}, \]  

(5.132)

and we assume this, although other choices are possible. In particular, (5.132) is not frame indifferent, but this is hardly of significance since the eddy viscosity itself is unreliable away from the surface. (We comment further on this in the notes at the end of the chapter.)

To convert to the constant eddy viscosity model of the preceding section, equation (5.68), we would rescale \( u, w, p \sim 1/\varepsilon \), and choose \( \eta = \varepsilon \); thus \( \varepsilon^2 = 1/R \).

5.7.5 Surface roughness layer

The basic shear flow near a flat surface \( z = 0 \) is given by (5.122), and in dimensionless terms is

\[ u = \frac{1}{\kappa} \ln z; \]

(5.133)

we will require similar behaviour when the flow is perturbed. Suppose, more generally, that as \( z \rightarrow \varepsilon s \),

\[ u \sim a + b \ln(z - \varepsilon s) + O(z - \varepsilon s), \]

(5.134)

which we shall find describes the solution away from the boundary. We put

\[ z = \varepsilon s + \nu Z, \]

(5.135)

where

\[ \nu = e^{-\kappa/\varepsilon}. \]

(5.136)

Additionally, we write

\[ u = -\frac{1}{\varepsilon} + U, \quad w = \varepsilon s_x U + \nu W, \quad \tau_1 = \varepsilon T_1, \quad \eta = \nu N. \]

(5.137)

Then we find that

\[ U_x + W_Z = 0, \]

\[ \frac{\partial \tau_3}{\partial Z} - \varepsilon^2 s_x \frac{\partial T_1}{\partial Z} + s_x \frac{\partial \rho}{\partial Z} \approx 0, \]

\[ \frac{\partial \rho}{\partial Z} \approx -\varepsilon^2 \left[ s_x \frac{\partial \tau_3}{\partial Z} + \frac{\partial T_1}{\partial Z} \right], \]

\[ N \approx \kappa^2 Z^2 \frac{\partial U}{\partial Z}, \]

\[ \tau_3 \approx N(1 - \varepsilon^2 s_x^2) \frac{\partial U}{\partial Z}, \]

\[ T_1 = -2 \kappa^2 s_x Z^2 \left( \frac{\partial U}{\partial Z} \right)^2, \]

(5.138)
where we have neglected transcendentally small terms proportional to $\nu$.

Correct to $O(\varepsilon^2)$, $\tau_3$ is constant through the roughness layer, and equal to its surface value $\tau$, and

$$\tau \approx \kappa^2 Z^2 \left( \frac{\partial U}{\partial Z} \right)^2,$$

again correct to $O(\varepsilon^2)$. The boundary conditions on $Z = 1$ (i.e., $z = \varepsilon s = z_0^* = \nu$) are $U = W = 0$, thus

$$U = \frac{\sqrt{\tau}}{\kappa} \ln Z,$$

and this must be matched to the outer solution (5.134). Rewriting (5.140) in terms of $u$ and $z$, we have

$$u \sim \frac{\sqrt{\tau} - 1}{\varepsilon} + \frac{\sqrt{\tau}}{\kappa} \ln(z - \varepsilon s),$$

and this is in fact the matching condition that we require from the outer solution. We see immediately that variations of $O(1)$ in $u$ yield small corrections of $O(\varepsilon)$ in $\tau$.

Solving for $W$, we have

$$W = -\left(\frac{\sqrt{\tau}}{\kappa} \right)' [Z \ln Z - Z],$$

where $(\sqrt{\tau})' = \partial \sqrt{\tau} / \partial x$, and in terms of $w$ and $z$, this is written

$$w = s_x + \varepsilon s_x u - \left(\frac{\sqrt{\tau}}{\kappa} \right)' \left[ \ln(z - \varepsilon s) - 1 + \frac{\kappa}{\varepsilon} \right] (z - \varepsilon s).$$

Hence the outer solution must satisfy (correct to $O(\varepsilon^2)$)

$$w \approx s_x + \varepsilon s_x u \text{ as } z \to \varepsilon s.$$  (5.144)

## 5.7.6 Outer solution

We turn now to the solution away from the roughness layer, in the presence of surface topography of amplitude $O(\varepsilon)$ and length scale $O(1)$. The topography has two effects. The $O(1)$ variation in length scale causes a perturbation on a height scale of $O(1)$, but the vertical displacement of the logarithmic layer by $O(\varepsilon)$ causes a shear layer of this thickness to occur. Thus the flow away from the surface consists of an outer layer of thickness $O(1)$, and an inner shear layer of thickness $O(\varepsilon)$. We begin with the outer layer.

We expand the variables as

$$u = u^{(0)} + \varepsilon u^{(1)} + \cdots,$$  (5.145)

etc., so that to leading order, from (5.128),

$$u_x^{(0)} + u_z^{(0)} = 0,$$
$$u_x^{(0)} + p_z^{(0)} = 0,$$
$$w_x^{(0)} + p_z^{(0)} = 0.$$  (5.146)
Notice that, at this leading order, the precise form of $\eta$ in (5.132) is irrelevant, as this outer problem is inviscid. We have

$$u^{(0)} + p^{(0)} = u_0(z),$$  \hspace{1cm} (5.147)

and

$$p_x^{(0)} = w_z^{(0)}, \quad p_z^{(0)} = -w_x^{(0)},$$  \hspace{1cm} (5.148)

which are the Cauchy-Riemann equations for $p^{(0)} + iw^{(0)}$, which is therefore an analytic function, and $p^{(0)}$ and $w^{(0)}$ both satisfy Laplace’s equation. The matching conditions as $z \to \varepsilon s$ can be linearised about $z = 0$, and if $w^{(0)} = w_0$ and $p^{(0)} = p_0$ on $z = \varepsilon s$, then from (5.144) we have

$$w^{(0)} = s_x \text{ on } z = 0. \hspace{1cm} (5.149)$$

Assuming also that $w^{(0)}, p^{(0)} \to 0$ as $z \to \infty$, we can write the solutions in the form

$$w^{(0)} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{zs_x d\xi}{[(x-\xi)^2 + z^2]}, \quad p^{(0)} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x-\xi) s_x d\xi}{[(x-\xi)^2 + z^2]},$$  \hspace{1cm} (5.150)

and in particular, $p^{(0)}$ on $z = \varepsilon s$ is given to leading order by $p_0$, where

$$p_0 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{s_x d\xi}{\xi - x} = H(s_x); \hspace{1cm} (5.151)$$

the integral takes the principal value, and $H$ denotes the Hilbert transform.

The shear velocity profile $u_0(z)$ is undetermined at this stage, although we would like it to be the basic shear flow profile; but to justify this, we need to go to the $O(\varepsilon)$ terms. At $O(\varepsilon)$, we have

$$u_x^{(1)} + w_x^{(1)} = 0, \hspace{1cm} u_z^{(1)} + p_z^{(1)} = \tau_{1x}^{(0)} + \tau_{3x}^{(0)} - \{u^{(0)} u_x^{(0)} + u^{(0)} u_z^{(0)}\}, \hspace{1cm}$$

$$w_x^{(1)} + p_x^{(1)} = \tau_{3x}^{(0)} - \tau_{1x}^{(0)} - \{w^{(0)} w_x^{(0)} + w^{(0)} w_z^{(0)}\}, \hspace{1cm}$$

$$\tau_3^{(0)} = \eta^{(0)}[u_x^{(0)} + u_z^{(0)}], \hspace{1cm}$$

$$\tau_1^{(0)} = 2\eta^{(0)}u_x^{(0)}, \hspace{1cm}$$

$$\eta^{(0)} = \kappa^2 z^2 \left| \frac{\partial u^{(0)}}{\partial z} \right|. \hspace{1cm} (5.152)$$

We can use the zero-th order solution to write (5.152)$_2$ in the form

$$u_x^{(1)} + p_x^{(1)} = \frac{\partial \tau_{3x}^{(0)}}{\partial z} + \frac{\partial}{\partial x} \left[ \tau_{1x}^{(0)} - \frac{1}{2}(u^{(0)} u_x^{(0)}) + u_0'(z) \psi^{(0)} \right], \hspace{1cm} (5.153)$$

where $\psi^{(0)}$ is the stream function such that $w^{(0)} = -\psi_x^{(0)}$, and specifically, we have

$$\psi^{(0)} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \ln[(x-\xi)^2 + z^2] p_0(\xi) d\xi, \hspace{1cm} (5.154)$$

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which can be found (as can the formulae in (5.150)) by using a suitable Green's function; (this is explained further below when we find \( p^{(1)} \)).

On integrating (5.153), we have to avoid secular terms which grow linearly in \( x \), and we therefore require the integral of the right hand side of (5.153) with respect to \( x \), from \(-\infty\) to \( \infty \), to be bounded. The integral of the derivative term is certainly bounded; thus the secularity condition requires \( \int_{-\infty}^{\infty} \tau_3^{(0)} \, dx \) to be bounded, and it is this condition that determines the function of integration \( u_0(z) \) in (5.147).

For the particular choice of \( \eta^{(0)} \) in (5.152), we have

\[
\eta^{(0)} = \kappa^2 z^2 (u'_0 + w_x^{(0)})
\]

(assumed positive), so that

\[
\tau_3^{(0)} = \kappa^2 z^2 (u'_0 + w_x^{(0)}) (u'_0 + 2w_x^{(0)}).
\]

The condition that \( \partial \tau_3^{(0)}/\partial z \) have zero mean is then

\[
\int_{-\infty}^{\infty} \frac{\partial}{\partial z} [\kappa^2 z^2 (u'_2 + 2w_x^{(0)2})] \, dx = 0,
\]

and thus \( \overline{\tau_3^{(0)}}/\partial z = 0 \), where the overbar denotes the horizontal mean. Thus (with \( \tau_3^{(0)} = 1 \) from the condition at \( z = \infty \)), \( u_0 \) is determined via

\[
u_0^2 + 2\overline{w_x^{(0)2}} = \frac{1}{\kappa^2 z^2}.
\]

The non-zero quantity \( 2\overline{w_x^{(0)2}} \) represents the form drag due to the surface topography. Note that the logarithmic behaviour of \( u_0 \) near \( z = 0 \) is unaffected by this extra term, and we can take

\[
u_0 = \frac{1}{\kappa} \ln z + O(z^2) \quad \text{as} \quad z \to 0.
\]

In particular, since \( p^{(0)} \approx p_0 + p_z^{(0)}|_{\varepsilon s}(z - \varepsilon s) \) as \( z \to \varepsilon s \), and \( p_z^{(0)}|_{\varepsilon s} = -w_x^{(0)}|_{\varepsilon s} = -s_{xx} \), we have

\[
u_0 \sim -p_0 + \frac{1}{\kappa} \ln z + s_{xx}(z - \varepsilon s) + O(z^2) \quad \text{as} \quad z \to \varepsilon s.
\]

From (5.156), we now have

\[
\tau_3^{(0)} = 1 + 3\kappa^2 z^2 u'_0 w_x^{(0)} + \frac{\partial \Phi}{\partial x},
\]

where we define

\[
\Phi = \int_{-\infty}^{x} 2\kappa^2 z^2 \left\{ w_x^{(0)2} - \overline{w_x^{(0)2}} \right\} \, dx.
\]

Hence from (5.153),

\[
u^{(1)} + p^{(1)} = \frac{\partial}{\partial z} \left[ 3\kappa^2 z^2 u'_0 w_x^{(0)} + \Phi \right] + \tau_1^{(0)} - \frac{1}{2} (u^{(0)2} + w^{(0)2}) + u'_0(z)\psi^{(0)} + u_1(z),
\]

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where \( u_1 \) must be determined at \( O(\varepsilon^2) \).

Now \( u_0 \sim \frac{1}{\kappa} \ln z + O(\varepsilon^2) \), \( \Phi = O(\varepsilon^2) \), \( \tau^{(0)}_1 = O(z) \), \( w^{(0)} = s_x + O(z) \), \( \psi^{(0)} = -s - z p_0 + O(\varepsilon^2) \) (this last follows from manipulation of (5.154)). Therefore, as \( z \to 0 \),

\[
\begin{align*}
\tau^{(1)} &= -p_{10} + 3\kappa s_x - \frac{1}{2} \left[ s_x^2 + \left\{ -p_0 + \frac{1}{\kappa} \ln z \right\}^2 \right] + \frac{1}{\kappa z} (s x - p_0) + u_1 + O(z), \quad (5.164)
\end{align*}
\]

where \( p_{10} = p^{(1)}|_{z=0} \).

### 5.7.7 Determination of \( p_{10} \)

Define the Green’s function

\[
K(x, z; \xi, \zeta) = -\frac{1}{4\pi} \left[ \ln\{(x - \xi)^2 + (z - \zeta)^2\} + \ln\{(x - \xi)^2 + (z + \zeta)^2\} \right]. \quad (5.165)
\]

We then have, for example,

\[
\begin{align*}
p^{(0)} &= \int \int_{\zeta > 0} \left[ K \nabla^2 p^{(0)} - p^{(0)} \nabla^2 K \right] d\xi d\zeta \\
&= \oint \left( K \frac{\partial p^{(0)}}{\partial n} - p^{(0)} \frac{\partial K}{\partial n} \right) ds \\
&= -\int_0^\infty K \frac{\partial p^{(0)}}{\partial \xi} d\xi = \int_0^\infty K \frac{\partial w^{(0)}}{\partial \xi} d\xi = -\int_0^\infty w^{(0)} \frac{\partial K}{\partial \xi} d\xi, \quad (5.166)
\end{align*}
\]

whence we derive (5.150) for example; the integrals with respect to \( \xi \) are taken along \( \zeta = 0 \).

Next, expanding (5.144) about \( z = 0 \), we find

\[
w^{(1)} \sim (s u^{(0)})_x \quad \text{as} \quad z \to 0. \quad (5.167)
\]

Putting

\[
w^{(1)} = s_x u_0 + W, \quad (5.168)
\]

we deduce the condition

\[
W = -(s p_0)_x \quad \text{on} \quad z = 0, \quad (5.169)
\]

and from (5.152)

\[
\begin{align*}
\tau^{(1)}_z - W_x &= R, \\
\tau^{(1)}_x + W_x &= S,
\end{align*}
\]

where

\[
\begin{align*}
R &= \tau^{(0)}_{1x} + \tau^{(0)}_{3z} - \{ u^{(0)} u^{(0)}_x + w^{(0)} u^{(0)}_z - s_x u^{(0)} \} , \\
S &= \tau^{(0)}_{3x} - \tau^{(0)}_{1z} - \{ u^{(0)} w^{(0)}_x + w^{(0)} w^{(0)}_z + s_{xx} u_0 \}.
\end{align*}
\]

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\[ \nabla^2 p^{(1)} = R_x + S_z, \quad (5.170) \]

and it follows from using the Green’s function as in (5.166) that, after some manipulation involving Green’s theorem,

\[ p^{(1)} = \int \int_{\zeta > 0} (RK_\zeta + SK_\zeta) \frac{d\xi d\zeta}{(x - \xi)^2 + \zeta^2} - \int_0^\infty K_\zeta W d\xi, \quad (5.171) \]

and therefore

\[ p_{10} = \frac{1}{\pi \int \int_{\zeta > 0} \frac{(x - \xi)R(\xi, \zeta) - \zeta S(\xi, \zeta)}{(x - \xi)^2 + \zeta^2} d\xi d\zeta - \frac{1}{\pi} \int_{-\infty}^\infty \frac{(sp_0)\xi d\xi}{\xi - x}. \quad (5.172) \]

### 5.7.8 Matching

Overall, then, the outer solution can be written, as \( z \to 0 \), in the form (using (5.160))

\[ u \sim -p_0 + \frac{1}{\kappa} \ln z + s_{xx}(z - \varepsilon s) + \varepsilon \left[ -p_{10} + 3\kappa s_x - \frac{1}{2}s_x^2 - \frac{1}{2}\varepsilon^2 + \frac{p_0}{\kappa} \ln z \right. \]

\[ \left. - \frac{1}{2\kappa^2} \ln^2 z - \frac{s}{\kappa z} - \frac{p_0}{\kappa} + A_1 \right]. \quad (5.173) \]

If we define

\[ \sqrt{\tau} = 1 + \varepsilon A_1 + \varepsilon^2 A_2 + \ldots, \quad (5.174) \]

then (5.141) takes the form

\[ u \sim A_1 + \frac{1}{\kappa} \ln z + \varepsilon \left[ -\frac{s}{\kappa z} + \frac{A_1}{\kappa} \ln z + A_2 \right] + \ldots, \quad (5.175) \]

and the leading order term can be matched directly to that of (5.173) by choosing

\[ A_1 = -p_0. \quad (5.176) \]

Using (5.176), (5.174) and (5.151), we have

\[ \tau \approx 1 + \frac{2\varepsilon}{\pi} \int_{-\infty}^\infty \frac{s_\xi d\xi}{x - \xi}, \quad (5.177) \]

and this can be compared with (5.103). Whereas in (5.103) \( K(x) = 0 \) for \( x < 0 \), the kernel \( K(x) \) in (5.177) is proportional to \( 1/x \) for all \( x \), and thus non-zero for \( x < 0 \). Consequently, there is no instability, and to find an instability we need to progress to the next order term.

Unfortunately, the \( O(\varepsilon) \) terms do not match because the terms \( \pm \frac{p_0}{\kappa} \ln z \) in the two expansions are not equal, and also because of the linear term in (5.173). In order to match the expansions to \( O(\varepsilon) \), we have to consider a further, intermediate layer: this is the shear layer we alluded to earlier.
5.7.9 Shear layer

A distinguished limit exists when \( z = O(\varepsilon) \), and thus we put

\[
\begin{align*}
  z &= \varepsilon s + \varepsilon \zeta, \quad w = s_x + \varepsilon [us_x + W], \\
  \eta &= \varepsilon N, \quad \tau_1 = \varepsilon T_1, \\
  u &= -p_0 + \frac{1}{\kappa} \ln(z - \varepsilon s) + \varepsilon v, \quad (5.178)
\end{align*}
\]

and from (5.173) and (5.141) (using (5.174) and (5.176)), we require

\[
\begin{align*}
  v &\sim s_{xx} \zeta - p_{10} + 3\kappa s_x - \frac{1}{2} s_x^2 - \frac{1}{2} p_0^2 - \frac{p_0}{\kappa} + \frac{p_0}{\kappa} \ln \varepsilon \zeta + \left( u_1 - \frac{1}{2\kappa^2} \ln^2 \varepsilon \zeta \right) \quad \text{as} \quad \zeta \to \infty, \\
  v &\sim A_2 - \frac{p_0}{\kappa} \ln \varepsilon \zeta \quad \text{as} \quad \zeta \to 0. \quad (5.179)
\end{align*}
\]

It follows from (5.178) that

\[
\begin{align*}
  N &= \kappa^2 \zeta^2 \frac{\partial u}{\partial \zeta}, \\
  T_1 &= 2N[u_x - s_x u_x], \\
  \tau_3 &= N[u_{xx} + \varepsilon s_{xx} + O(\varepsilon^2)], \\
  u_x + W_{\zeta} &= 0, \\
  (u + p)_x - s_x p_{\zeta} &= \tau_{3\zeta} - \varepsilon [uu_x + Wu_{xx}] + O(\varepsilon^2), \\
  p_{\zeta} &= -\varepsilon s_{xx} + O(\varepsilon^2). \quad (5.180)
\end{align*}
\]

Since we have \( p = p_0 + \varepsilon p_{10} \) and \( W = 0 \) on \( \zeta = 0 \), then

\[
\begin{align*}
  p &= p_0 + \varepsilon (p_{10} - s_{xx} \zeta) + O(\varepsilon^2), \\
  W &= p_0' \zeta + O(\varepsilon), \quad (5.181)
\end{align*}
\]

and thus \( v \) satisfies

\[
\begin{align*}
  v_x + p_{10}' s_{xx} + s_x s_{xx} &= \frac{\partial}{\partial \zeta} [2\kappa \zeta v_{\zeta} + \kappa \zeta s_{xx}] - \left[ -p_0' \left( -p_0 + \frac{1}{\kappa} \ln \varepsilon \zeta \right) + \frac{p_0'}{\kappa} \right] + O(\varepsilon), \quad (5.182)
\end{align*}
\]

together with (5.179).

The solution of (5.182) is

\[
  v \approx -p_{10} - \frac{1}{2} s_x^2 - \frac{p_0}{\kappa} - \frac{1}{2} p_0^2 + \frac{p_0}{\kappa} \ln \varepsilon \zeta + s_{xx} \zeta + 3\kappa s_x + V, \quad (5.183)
\]

where

\[
  \frac{\partial V}{\partial x} = \frac{\partial}{\partial \zeta} \left[ 2\kappa \zeta \frac{\partial V}{\partial \zeta} \right], \quad (5.184)
\]

and (5.179) implies

\[
  V \to 0 \quad \text{as} \quad \zeta \to \infty,
\]

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\[ V \sim A_2^* - \frac{2p_0}{\kappa} \ln \varepsilon \zeta \quad \text{as} \quad \zeta \to 0, \]  
(5.185)

where
\[ A_2 = A_2^* - p_{10} - \frac{1}{2}s_x^2 - \frac{p_0}{\kappa} - \frac{1}{2}p_0^2 + 3\kappa s_x. \]  
(5.186)

The solution of (5.184) which tends to zero as \( \zeta \to \infty \) is
\[ V = \int_{-\infty}^{\infty} \hat{V}(\zeta, k)e^{ikx} \, dk, \]  
(5.187)

where the Fourier transform \( \hat{V} \) (as thus defined) is given by
\[ \hat{V} = BK_0 \left[ \left( \frac{2ik\zeta}{\kappa} \right)^{1/2} \right], \]  
(5.188)

the square root is chosen so that Re \((ik)^{1/2} > 0\), and \( K_0 \) is a modified Bessel function of order zero. Evidently we require
\[ \hat{V} \sim \hat{A}_2^* - \frac{2p_0}{\kappa} \ln (\varepsilon \zeta) \quad \text{as} \quad \zeta \to 0, \]  
(5.189)

where the overhat defines the Fourier transform, in analogy to (5.187). Now \( K_0(\xi) \sim -\ln \frac{1}{2}\xi - \gamma \) as \( \xi \to 0 \), where \( \gamma \approx 0.5772 \) is the Euler-Mascheroni constant. Also
\[ \left( \frac{2ik\zeta}{\kappa} \right)^{1/2} = \left( \frac{2|k|\zeta}{\kappa} \right)^{1/2} \exp \left[ \frac{i\pi}{4} \text{sgn} k \right]; \]  
(5.190)

therefore (5.188) implies
\[ \hat{V} \sim -B \left[ \gamma + \frac{1}{2} \ln |k| - \frac{1}{2} \ln 2\kappa + \frac{1}{2} \ln \zeta + \frac{i\pi}{4} \frac{k}{|k|} \right], \]  
(5.191)

and matching this to (5.189) implies
\[ B = \frac{4\hat{p}_0}{\kappa}, \]  
(5.192)

whence
\[ \hat{A}_2^* = \frac{2\hat{p}_0}{\kappa} \ln \varepsilon - \frac{4\hat{p}_0}{\kappa} \left[ \gamma + \frac{1}{2} \ln |k| - \frac{1}{2} \ln 2\kappa + \frac{i\pi k}{4|k|} \right]. \]  
(5.193)

We have \( \hat{s}_x = iks, \quad \hat{H}(s_x) = -|k|\hat{s} \), and \( \hat{J} \ast \hat{s}_x = |k|\hat{s} \ln |k| \), where \( J \ast s_x \) is the convolution of \( J \) with \( s_x \), and \( \hat{J} = -(i/2\pi) \ln |k| \text{sgn} k \). (The convolution theorem here takes the form \( \hat{f} \ast g = 2\pi \hat{f} \hat{g} \).) It follows from this that
\[ J(x) = -\frac{1}{\pi x} \left[ \gamma + \ln |x| \right]. \]  
(5.194)

\(^7\)Assuming the principal branch of the square root, this implies we take \( k = |k|e^{-i\pi} \) when \( k \) is negative.
Thus
\[ A_2 = \frac{2}{\kappa} \left( \ln 2 \varepsilon \kappa - 2 \gamma \right) p_0 + \frac{\pi}{\kappa} s_x + \frac{1}{\pi \kappa} J * s_x, \quad (5.195) \]
and, from (5.186),
\[ A_2 = \frac{2}{\kappa} \left( \ln 2 \varepsilon \kappa - 2 \gamma - \frac{1}{2} \right) p_0 + \left( \frac{\pi}{\kappa} + 3\kappa \right) s_x 
+ \frac{1}{\pi \kappa} J * s_x - p_{10} - \frac{1}{2} p_0^2 - \frac{\gamma^2}{2}, \quad (5.196) \]
where \( J \) is given by (5.194), \( p_0 = H(s_x) \) ((5.151)), and \( p_{10} \) is given by (5.172).

We can summarise our calculation of the basal shear stress as follows. From (5.174), (5.176) and (5.151) we have
\[ \tau = 1 + \varepsilon B_1 + \varepsilon^2 B_2 + \ldots, \quad (5.197) \]
where
\[ B_1 = 2A_1 = -2H(s_x), \quad B_2 = 2A_2 + A_1^2. \quad (5.198) \]

Using (5.186) and (5.193), we find after a little algebra that the transform of \( B_2 \) is
\[ \hat{B}_2 = \frac{\hat{B}_1}{\kappa} \left[ -2 \ln 2 \varepsilon \kappa + 2 \ln |k| + i \pi \text{sgn} k + 4 \gamma + 1 \right] + \hat{C}, \quad (5.199) \]
where \( \hat{C} \) is the transform of \( C = -2p_{10} - s_x^2 + 6\kappa s_x. \quad (5.200) \)

### 5.7.10 Linear stability

The Exner equation is, in appropriate dimensionless form,\(^8\)
\[ \varepsilon s_t + q_x = 0, \quad (5.201) \]
and since \( q = q(\tau), \)
\[ q = q_1 - 2\varepsilon q_1' p_0 + \varepsilon^2 \left[ (2A_2 + p_0^2) q_1' + 2p_0^2 q_1'' \right] + \ldots, \quad (5.202) \]
where \( q_1 = q(1), q_1' = q'(1), q_1'' = q''(1). \)

Thus \( s \) satisfies the nonlinear evolution equation
\[ \frac{\partial s}{\partial t} - 2q_1 \frac{\partial p_0}{\partial x} + \varepsilon \frac{\partial}{\partial x} \left[ (2A_2 + p_0^2) q_1' + 2p_0^2 q_1'' \right] \approx 0. \quad (5.203) \]
This is
\[ \frac{\partial s}{\partial t} - \alpha \frac{\partial p_0}{\partial x} + \varepsilon \frac{\partial}{\partial x} \left[ q_1' \left( 2\omega s_x + 2s_x \right) + 2q_1'' \right] = 0, \]
\(^8\)Note that the definition of \( \varepsilon \) here is that pertaining to the mixing length theory, i.e., (5.121) and not (5.48).
where

\[
p_0 = H(s_x), \quad (5.204)
\]

\[
\alpha = 2q_1' \left[ 1 - \frac{2\varepsilon}{\kappa} \left( \ln 2\varepsilon\kappa - 2\gamma - \frac{1}{2} \right) \right],
\]

\[
\omega = \frac{\pi}{\kappa} + 3\kappa,
\]

\[
\lambda = \frac{1}{\pi\kappa}. \quad (5.205)
\]

We linearise (5.204) for small \( s \) by neglecting the terms in \( s_x^2 \) and \( p_0^2 \). Taking the Fourier transform (as defined here in (5.187)), we have

\[
\hat{s}_t = i\kappa\alpha\hat{p}_0 - i\varepsilon q_1' \left( 2\omega ik\hat{s} + 4\pi\lambda ik\hat{J}s - 2\hat{p}_{10} \right). \quad (5.206)
\]

From (5.172),

\[
p_{10} = \int_0^\infty (a \ast R + b \ast S) d\zeta - H\{ (sp_0)_x \}, \quad (5.207)
\]

where

\[
a(x, \zeta) = \frac{x}{\pi(x^2 + \zeta^2)}, \quad b(x, \zeta) = -\frac{\zeta}{\pi(x^2 + \zeta^2)}. \quad (5.208)
\]

Hence, neglecting the quadratic Hilbert transform term,

\[
\hat{p}_{10} = 2\pi \int_0^\infty (\hat{a}\hat{R} + \hat{b}\hat{S})d\zeta. \quad (5.209)
\]

Calculation of \( \hat{a} \) and \( \hat{b} \) gives

\[
\hat{a} = -\frac{i}{2\pi} e^{-|k|\zeta} \text{sgn } k, \quad \hat{b} = -\frac{1}{2\pi} e^{-|k|\zeta}, \quad (5.210)
\]

so that

\[
\hat{p}_{10} = -\int_0^\infty [i\hat{R} \text{sgn } k + \hat{S}] e^{-|k|\zeta} d\zeta. \quad (5.211)
\]

Now

\[
\tau_3^{(0)} = 1 + 3\kappa z u_x^{(0)} + 2\kappa^2 z^2 w_x^{(0)2},
\]

\[
\tau_1^{(0)} = -2\kappa z u_x^{(0)} - 2\kappa^2 z^2 w_x^{(0)} u_x^{(0)},
\]

\[
u^{(0)} = u_0(z) - p^{(0)},
\]

\[
u_x^{(0)} = -u_x^{(0)},
\]

\[
u_z^{(0)} = u_z^{(0)} + w_x^{(0)}, \quad (5.212)
\]

thus, retaining only the perturbed linear (in \( s \)) terms, we have from (5.170)

\[
\hat{R} \approx ik\hat{t}_3 + \hat{t}_3 z + u_0 \hat{w}_z - u_0' [\hat{w} - ik\hat{s}],
\]

\[
\hat{S} \approx ik\hat{t}_3 - \hat{t}_1 z - iku_0 [\hat{w} + ik\hat{s}], \quad (5.213)
\]

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where \( \hat{w} = \hat{w}^{(0)} \), and
\[
\hat{t}_1 = -2i \kappa z \hat{p}, \quad \hat{t}_3 = 3i \kappa z \hat{w},
\]
where \( \hat{p} = \hat{p}^{(0)} \).

Finally, from (5.150),
\[
w^{(0)} = -b(x, z) \ast s_x, \quad p^{(0)} = -a(x, z) \ast s_x,
\]
whence using (5.210),
\[
\hat{w} = ik \hat{s} e^{-|k|z}, \\
\hat{p} = -|k| \hat{s} e^{-|k|z},
\]
and we eventually obtain
\[
\hat{p}_{10} = -\hat{s} \int_0^\infty \left[ k^2 u_0 (1 + 2 e^{-|k|\zeta}) - |k| u_0' (1 - e^{-|k|\zeta}) - 5 i \kappa k |k| e^{-|k|\zeta} \right] e^{-|k|\zeta} d\zeta.
\]

Simplification of this, using the fact that \( \int_0^\infty e^{-t} \ln t dt = -\gamma \), where \( \gamma \approx 0.5772 \) is the Euler-Mascheroni constant, yields
\[
\hat{p}_{10} = \hat{s} \left[ \frac{2|k|}{\kappa} (\ln 2|k| + \gamma) + \frac{5}{2} i \kappa k \right].
\]

Solutions of (5.206) are \( \hat{s} = e^{\sigma t} \), where \( \sigma = r - i c \), and after some simplification, we find that the growth rate \( r \) is
\[
r = 2k^2 \varepsilon q_1 \left( \frac{\pi}{\kappa} + \frac{1}{2} \kappa \right),
\]
and the wave speed \( c \) is
\[
c = 2q_1 |k| \left[ 1 + \frac{2 \varepsilon}{\kappa} \left\{ -\left( 1 - \frac{1}{2\pi} \right) \ln |k| - \ln 4 \varepsilon \kappa + \gamma + \frac{1}{2} \right\} \right].
\]

Thus dunes grow, as \( r > 0 \), on a time scale of \( O(1/\varepsilon) \), while the waveforms move downstream at a speed \( c \approx 2q_1 |k| = O(1) \).

This apparently more realistic theory for dune-forming instability is less satisfactory than the constant eddy viscosity theory, because the growth rate \( r \propto k^2 \), and the basic model is again ill-posed. As before, we can stabilise the model by including the downslope force, thus replacing the stress by the effective stress defined using (5.109). The effect of this is to add a term to the stress definition in (5.174), which can then be written as
\[
\tau_e = 1 - 2 \varepsilon p_0 + 2 \varepsilon^2 (A_2 + \ldots) - \hat{\beta} s_x,
\]
where the definition of \( \hat{\beta} \) differs from that in (5.111) because of the different scaling used in the aeolian model. Using (5.124), \( x \sim d \) and \( s \sim \varepsilon d \), we find
\[
\hat{\beta} = \frac{\Delta \rho}{\rho} \frac{D_s}{d} \frac{1}{\varepsilon F^2},
\]
where \( \Delta \rho \) is the density difference between the two fluids.
where the Froude number is

\[ F = \frac{U_\infty}{\sqrt{gd}} \]  

(5.223)

Using values \( \Delta \rho/\rho = 2.6 \times 10^3 \), \( D_s/d = 10^{-6} \), \( \varepsilon \sim 0.03 \), \( F^2 \sim 0.04 \) (based on \( d = 1000 \) m and \( U_\infty = 20 \) m s\(^{-1}\)), we find \( \hat{\beta} \sim 2.2 \).

If we consult (5.196), we see that the destabilising term arises from that proportional to \( s_x \) in \( A_2 \). Effectively we can write

\[ \tau_e = 1 + \ldots + \left[ \varepsilon^2 \left( \frac{2\pi}{\kappa} + \kappa \right) - \hat{\beta} \right] \frac{\partial s}{\partial x} + \ldots, \]  

(5.224)

where the modification of the coefficient \( \omega \) reflects the effect of the terms in \( J \) and \( p_{10} \), as indicated by (5.219). We see that the downslope term stabilises the system if \( \hat{\beta} > O(\varepsilon^2) \), and thus practically if \( F^2 < 1 \). On the Earth, a typical value is \( F^2 = 0.04 \), so that the instability is removed, at all wave numbers. This is distinct from the constant eddy viscosity case, because the stabilising term has the same wave number dependence as the destabilising one.

If we ignore the stabilising term in \( \hat{\beta} \), then the situation is somewhat similar to the rill-forming instability which we will study in chapter 6. There the instability is regularised at long wavelength by inclusion of singularly perturbed terms. The most obvious modification to make here in a similar direction is to allow for a finite thickness of the moving sand layer. It seems likely that this will make a substantial difference, because the detail of the mixing length model relies ultimately on the existence of an exponentially small roughness layer through which the wind speed drops to zero. It is noteworthy that the constant eddy viscosity model does not share this facet of the problem.

### 5.8 Separation at the wave crest

The constant eddy viscosity model can produce a genuine instability, with decay at large wave numbers. If pushed to a nonlinear régime, it allows shock formation, although it also allows unlimited wavelength growth. The presumably more accurate mixing length theory actually fares somewhat worse. It can produce a very slow instability via an effective negative diffusivity, but this is easily stabilised by downslope drift. It is possibly the case that specific consideration of the mobile sand layer will alleviate this result.

A complication arises at this point. Aeolian sand dunes inevitably form slip faces. There is a jump in slope at the top of the slip face, and the wind flow separates, forming a wake (or cavity, or bubble). One authority is of the opinion that no model can be realistic unless it includes a consideration of separation. In this section we will consider a model which is able to do this. Before doing so, it is instructive to consider how such separation arises.

If the constant eddy viscosity model has any validity, it suggests that the uniform flat bed is unstable, and that travelling waves grow to form shocks. If the slope within the shocks is steep enough to exceed the angle of repose of sand grains (some 34°),
then a slip face will occur, with the sand resting on the slip face at this angle. The
turbulent flow over the dune inevitably separates at the cusp of the dune, forming a
separation bubble, as indicated in figure 5.13. The formation of a separation bubble
makes the model fundamentally nonlinear, and it provides a possible mechanism for
length scale selection. It is thus an attractive possible way out of the conundrums
concerning instability alluded to above.

![Diagram](image)

Figure 5.13: Separation behind a dune.

It is simplest to treat the separation bubble in the context of the mixing length
theory, and this we now do, despite our misgivings about its applicability for small
amplitude perturbations. We suppose that there is a periodic sequence of dunes, with
period chosen to be $2\pi$. We suppose that there is a slip face, as shown in figure 5.13,
and we suppose the corresponding separation bubble occupies the interval $(a, b)$. We
denote the bubble interval as $B$, and the corresponding attached flow region as $B'$.

Because our method will use complex variables, it is convenient to rechristen the
space coordinates as $x$ and $y$, and the corresponding velocity components as $u$ and $v$.
At leading order, the inviscid flow is described by the outer equations (5.146):

$$
\begin{align*}
    u_x + v_y &= 0, \\
    u_x + p_y &= 0, \\
    v_x + p_y &= 0, \\
\end{align*}
$$

(5.225)

and these are valid in $y > \varepsilon s$. From these it follows that $p$ and $v$ satisfy the Cauchy-
Riemann equations, and thus

$$
p + iv = f(z)
$$

(5.226)

is an analytic function, where $z = x + iy$.

The boundary conditions for $p$ and $v$ are that both tend to zero as $y \to \infty$, and
$v$ satisfies the no flow through condition (5.144), $v = s_x + \varepsilon u s_x$ on $y = \varepsilon s$. These
completely specify the problem in the absence of a separation bubble.

If we suppose that a separation bubble occurs, as shown in figure 5.13, then its
upper boundary is unknown, and must be determined by an extra boundary condition.
We let $y = \varepsilon s(x)$ denote this unknown upper boundary, and define the ground surface
to be $y = \varepsilon s_0(x)$; thus $s(x) = s_0(x)$ for $x \in B'$.
There are various ways to provide the extra condition. Two such are that the pressure, or alternatively the vorticity, are constant in the bubble. We shall suppose the former, and therefore we prescribe

\[ p = p_B \quad \text{for} \quad y = \varepsilon s, \quad x \in B. \]

(5.227)

The bubble pressure \( p_B \) is an unknown constant, and must be determined as part of the solution.

Separation occurs because the viscous boundary layer (here, the roughness layer) detaches from the surface, forming a free shear layer at the top of the bubble, which rapidly thickens to form a more diffuse upper boundary. The assumption of constant pressure in the bubble is essentially a consequence of this shear layer, implying that mean fluid velocities in the bubble are small.

For small \( \varepsilon \), we can expand the boundary conditions at \( y = \varepsilon s \) about \( y = 0 \), so that to leading order, the problem becomes that of finding an analytic function \( f(z) = p + iv \) in the upper half plane \( \text{Im} \ z > 0 \), satisfying

\[
\begin{align*}
    f & \to 0 \quad \text{as} \quad z \to \infty, \\
    v & = s_x \quad \text{on} \quad y = 0, \\
    p & = p_B \quad \text{on} \quad y = 0, \quad x \in B.
\end{align*}
\]

(5.228)

The extra pressure condition should help determine \( s \) in \( B \), but the endpoint locations are not necessarily known. Specification of the behaviour of the solution at the endpoints is necessary to determine these. Firstly, we expect \( s \) to be continuous at the end points:

\[ s(a) = s_0(a), \quad s(b) = s_0(b). \]

(5.229)

A difference now arises depending on whether a slip face occurs or not. If not, then the bed slope is continuous, and at the upstream end point \( x = a \), we might surmise that boundary layer separation is associated with the skin friction dropping to zero. Now from (5.174) and (5.176), we have the surface stress defined by

\[ \sqrt{\tau} = 1 - \varepsilon \rho_0, \]

(5.230)

where \( \rho_0 \) is the surface pressure. The only apparent interpretation of this which we can make in our simplified model is to require that

\[ p \to +\infty \quad \text{on} \quad y = 0 \quad \text{as} \quad x \to a- \in B'; \]

(5.231)

more detailed consideration of the boundary layer structure near the separation point would be necessary to be more precise than this. We do not pursue this possibility here, mainly because the more relevant situation is when a slip face is present.

If we suppose a slip face is present, then we can presume that separation occurs at its top, and this determines the point \( x = a \). In addition, it is natural to suppose that boundary layer detachment occurs smoothly, in the sense that we suppose the slope of \( s \) is continuous at \( a \):

\[ s'(a+) = s'_0(a-); \]

(5.232)
this implies that $v$ is continuous at $x = a$. If possible, we would like to have smooth reattachment at $b$, and in addition (and in fact, because of this) continuity of pressure also:

$$[p]_{b}^{b+} = [p]_{a}^{a+} = 0, \quad s'(b-) = s'_0(b+). \quad (5.233)$$

We shall in fact find that all these conditions can be satisfied. This is not always the case in such problems, and sometimes (worse) singularities have to be tolerated. The choice of the behaviour of the solution at the end points actually constitutes the most subtle part of solving Hilbert problems.

### 5.8.1 Formulation of Hilbert problem

The first thing we do is to analytically continue $f(z)$ into the lower half plane. Specifically, we define

$$G(z) = \begin{cases} 
\frac{1}{2} \left[ f(z) - p_B \right], & \text{Im } z > 0, \\
-\frac{1}{2} \left[ \bar{f}(z) - p_B \right], & \text{Im } z < 0. 
\end{cases} \quad (5.234)$$

$G$ is analytic in both the upper and lower half planes, and if $G_+$ and $G_-$ denote the limiting values of $G$ as $z \to x$ from above and below, then

$$G_+ + G_- = is', \quad G_+ - G_- = p - p_B, \quad (5.235)$$

everywhere on the real axis.

Because of the assumed periodicity in $x$, we make the following transformations:

$$\zeta = e^{iz}, \quad \xi = e^{ix}, \quad G(z) = H(\xi). \quad (5.236)$$

The geometry of the problem is then illustrated in figure 5.14. The problem to solve is identical to (5.235), replacing $G$ by $H$, and thus we have the standard Hilbert problem

$$H_+ - H_- = 0 \quad \text{on } B, \quad H_+ + H_- = i\sigma_0 \quad \text{on } B', \quad (5.237)$$

where $\sigma_0(\xi) = s_0'(x)$. We have to solve this subject to the supplementary conditions

$$H(0) = -\frac{1}{2}p_B, \quad H(\infty) = \frac{1}{2}p_B; \quad (5.238)$$

the first of these in fact implies the second automatically. We seek to apply the conditions that both $\frac{1}{2}(p - p_B) = \text{Re } H$ and $\frac{1}{2}v = \text{Im } H$ are continuous (thus $H$ is continuous) at both endpoints $\xi = \xi_a = e^{i\alpha}$ and $\xi = \xi_b = e^{ib}$. Given $H$ satisfying (5.237), then the separation bubble boundary is given by the solution of

$$s' = -2iH \quad \text{on } B, \quad s(a) = s_0(a), \quad (5.239)$$

and the pressure on $B'$ is given by

$$p = p_B + H_+ - H_- \quad \text{on } B'. \quad (5.240)$$
Figure 5.14: $B$ and $B'$ on the unit circle in the complex $\zeta$ plane. $B'$ is a branch cut for the solution of the Hilbert problem (5.237).

**Solution**

The solution to (5.237), given the location of $a$ and $b$, is as follows. Define a function $\chi(\zeta)$ such that

$$\chi_+ + \chi_- = 0 \quad \text{on} \quad B'$$

(and $\chi$ is analytic away from $B'$); then

$$\left( \frac{H}{\chi} \right)_+ - \left( \frac{H}{\chi} \right)_- = \frac{i\sigma_0}{\chi_+},$$

and by the discontinuity theorem, we have

$$H = \chi(\zeta) \int_{B'} \frac{i\sigma_0(t)\,dt}{\chi_+(t)(t - \zeta)} + \chi P,$$

where $P$ is an as yet undetermined polynomial. To find $P$, we must specify $\chi$, and this in turn depends on the required singularity structure of the solution.

The smoothness of $H$ is essentially that of $\chi$, and so we will choose the function

$$\chi = [(\zeta - \xi_a)(\zeta - \xi_b)]^{1/2}.$$  (5.244)

The most general choice is $\chi = (\zeta - \xi_a)^{m_a + \frac{1}{2}}(\zeta - \xi_b)^{m_b + \frac{1}{2}}$, where $m_a$ and $m_b$ are integers, but most of these possibilities can in general be eliminated by requirements either of continuity or at least integrability of the solution.

We consider the behaviour of the Cauchy integral

$$\Phi(\zeta) = \frac{1}{2\pi i} \int_{B'} \frac{\phi(t)\,dt}{t - \zeta}$$

(5.245)
near the end points of integration. Note that in the present case,

$$\phi(t) = \frac{i\sigma_0(t)}{\chi_+(t)}.$$  

(5.246)

First suppose that $\phi(t)$ is continuous at an end point.\(^9\) Then we have

$$\Phi(\zeta) = \pm \frac{\phi(c)}{2\pi i} \ln(\zeta - c) + O(1),$$  

(5.247)

where $c$ denotes either end point of $B'$, and the upper and lower signs apply at the right ($\xi_a$) and left ($\xi_b$) hand ends of $B'$, respectively. (5.247) applies as $\zeta \to c$, with $\zeta \notin B'$.

Similarly, for $\xi \in B'$,

$$\Phi(\xi) = \pm \frac{\phi(c)}{2\pi i} \ln(\xi - c) + O(1),$$  

(5.248)

where $\Phi(\xi)$ denotes the principal value of the integral (and $\Phi(\xi) = \frac{1}{2} [\Phi_+(\xi) + \Phi_-(\xi)]$).

Bearing in mind (5.246), we see that if $\chi$ is unbounded at $c$, and specifically goes algebraically to infinity, then the corresponding Cauchy integral is bounded, and thus $H$ will be unbounded (unless the choice of $P$ can be chosen to remove the singularity).

Using the definition

$$H = \chi(\zeta) [\Phi(\zeta) + P],$$  

(5.249)

we have from (5.239) and (5.240) that

$$s' = -2iH(\xi) = -2i\chi(\xi) [\Phi(\xi) + P] \text{ on } B,$$

$$p - p_B = 2\chi(\xi) [\Phi(\xi) + P] \text{ on } B'. $$  

(5.250)

The implication of this is that if $\chi$ is unbounded at an end point, then in general both $p$ and $s'$ will also be unbounded, unless the choice of $P$ removes the singularity. The worst singularity we can tolerate is an integrable one, thus $\chi \sim (\zeta - c)^{-1/2}$.

Now suppose that $\chi$ is bounded at an end point, and specifically $\chi \sim (\zeta - c)^{1/2}$. (Any higher power causes the Cauchy integral to be undefined, because then $\phi$ is not integrable.) If we define $\tilde{\phi}$ via

$$\phi(t) \sim \frac{\tilde{\phi}(t)}{(t - c)^{1/2}} \text{ as } t \to c,$$  

(5.251)

then

$$\Phi(\zeta) = \frac{\tilde{\phi}(c)}{2(\zeta - c)^{1/2}} + o\left(\frac{1}{(\zeta - c)^{1/2}}\right), \quad \zeta \in B;$$

$$\Phi(\xi) = o\left(\frac{1}{(\xi - c)^{1/2}}\right), \quad \xi \in B'.$$  

(5.252)

\(^9\)More precisely, $\phi$ should be Hölder continuous, that is $|\phi(t_1) - \phi(t_2)| < K|t_1 - t_2|^{\gamma}$, for some positive $\gamma$.  

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It then follows from (5.250) that \( s' \) is bounded (and in fact continuous) and \( p \) is continuous at \( c \). It is because of this that we choose \( \chi \) as defined in (5.244), in order to satisfy the smoothness conditions (5.232) and (5.233).

In this case, the polynomial \( P \) must be zero in order to satisfy the condition at \( \zeta = \infty \), and we have
\[
H = \frac{\chi(\zeta)}{2\pi i} \int_{B'} \frac{i\sigma_0(t) dt}{\chi_+(t)(t - \zeta)}.
\]
We define the integrals
\[
I_0 = \frac{\chi_0}{2\pi i} \int_{B'} \frac{i\sigma_0(t) dt}{t\chi_+(t)}, \quad I_\infty = \frac{1}{2\pi i} \int_{B'} \frac{i\sigma_0(t) dt}{\chi_+(t)};
\]
we thus have \( H(0) = I_0, \quad H(\infty) = -I_\infty \), and the conditions in (5.238) correspond to prescribing
\[
I_0 = I_\infty = -\frac{1}{2}p_B.
\]
It is a straightforward exercise in contour integration to show that \( I_0 = I_\infty \), where the overbar denotes the complex conjugate, therefore (5.255) is tantamount to the single condition \( I_0 = -\frac{1}{2}p_B \). Because this is a complex-valued integral, (5.255) actually comprises two conditions for the two unknown quantities \( p_B \) and \( b \).

It remains to be seen whether \( s \) is continuous at \( b \). Since (5.255) determines \( b \), and \( s \) is fully determined by (5.239), it is not obvious that this will be the case. (If it were not, we would have to allow for a singularity in the solution at one of the end points.)

In fact, it is easy to show that (5.255) automatically implies that \( s \) is continuous at \( b \). To show this, it is sufficient to show that \( s \) is continuous over the periodic domain \([0, 2\pi]\). Equivalently, we need to show that
\[
I = \int_0^{2\pi} s' \, dx = \int_{B \cup B'} -i(H_+ + H_-) \frac{d\xi}{\xi} = 0,
\]
using (5.239) and (5.237). Denoting contours just inside and outside the unit circle as \( C_+ \) and \( C_- \) (see figure 5.15), we see that
\[
I = -\left[ \int_{C_+} \frac{H d\xi}{\xi} + \int_{C_-} \frac{H d\xi}{\xi} \right].
\]
\( H \) is analytic inside and outside the unit circle. The integral over \( C_+ \) is thus just \( 2\pi i H(0) \) using the residue theorem, while the integral over \( C_- \) can be extended by deforming the contour out to infinity, whence we obtain the integral \( 2\pi i H(\infty) \). Thus
\[
I = -2\pi i [I_0 - I_\infty] = 0,
\]
and continuity of \( s \) at \( b \) is assured. We have thus obtained a solution in which the separated streamline leaves and rejoins the surface smoothly, and the pressure is continuous at the end points.
Figure 5.15: The contours $C_+$ and $C_-$ lie just inside and outside the unit circle, respectively.

### 5.8.2 Calculation of the free boundary

In order to solve (5.239) for $s$, we need to evaluate $H$ on $B'$. There are various ways to do this. One simple one, which may be convenient for subsequent evolution of the bed using spectral methods, is to use a Fourier series representation. Let us suppose that

$$s(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx}, \quad (5.259)$$

so that

$$i\sigma_0(\xi) = \sum_{k=-\infty}^{\infty} d_k \xi^k, \quad (5.260)$$

where

$$d_k = -ka_k. \quad (5.261)$$

We suppose that the Laurent expansion for $i\sigma_0$ extends to the complex plane as an analytic function with singularities only at 0 and $\infty$. (This is automatically true for any finite such series.) Then we can write the solution for $H$ as

$$H = \frac{1}{2} [i\sigma_0(\zeta) - q(\zeta)\chi(\zeta)], \quad (5.262)$$

where $q$ has a Laurent expansion

$$q = \sum_{k=-\infty}^{\infty} l_k \zeta^k. \quad (5.263)$$

Then we obtain $s$ by solving

$$s' = s'_0 + iq\chi \quad (5.264)$$

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on \([a, b]\), with \(s(a) = s_0(a)\). In practice, we would obtain \(b\) by shooting.

Suppose that
\[
\frac{1}{\chi(\zeta)} = \sum_{r=0}^{\infty} \frac{f_r}{\zeta^{r+1}}, \quad |\zeta| > 1
\]  
(5.265)

(see question 5.10 for one way to calculate the coefficients); then we can write
\[
\frac{H}{\frac{\chi}{2}} = \sum_{m=-\infty}^{\infty} d_m \zeta^m \sum_{r=0}^{\infty} \frac{f_r}{\zeta^{r+1}} - \sum_{j=-\infty}^{\infty} l_j \zeta^j.
\]  
(5.266)

As \(\zeta \to \infty, \chi \sim \zeta\) and \(H \to \frac{1}{2}p_B\); equating coefficients of \(\zeta^j\) in (5.266) for \(j \geq 0\) yields
\[
l_j = \sum_{r=0}^{\infty} d_{j+r+1} f_r, \quad j \geq 0,
\]  
(5.267)

and for \(j = 0\) we have
\[
p_B = \sum_{r=0}^{\infty} d_r f_r - l_{-1}.
\]  
(5.268)

For \(|\zeta| < 1\), we find
\[
\frac{1}{\chi(\zeta)} = \frac{1}{\chi_0} \sum_{r=0}^{\infty} \bar{f}_r \zeta^r,
\]  
(5.269)

and thus
\[
\frac{H}{\frac{\chi}{2}} = \frac{1}{\chi_0} \sum_{r=0}^{\infty} \bar{f}_r \zeta^r \sum_{m=-\infty}^{\infty} d_m \zeta^m - \sum_{j=-\infty}^{\infty} l_j \zeta^j.
\]  
(5.270)

As \(\zeta \to 0, H \to -\frac{1}{2}p_B\); equating powers of \(\zeta^j\) for \(j \leq -1\), we find
\[
l_j = \frac{1}{\chi_0} \sum_{r=0}^{\infty} \bar{f}_r d_{j-r}, \quad j \leq -1,
\]  
(5.271)

and for \(j = 0\) we have
\[
p_B = \frac{1}{\chi_0} \sum_{r=0}^{\infty} \bar{f}_r d_{-r} - l_0.
\]  
(5.272)

Putting these results together, we find that (5.268) and (5.272) together give
(bearing in mind that \(d_{-k} = -\bar{d}_k\))
\[
p_B = \sum_{r=0}^{\infty} d_r f_r + \chi_0 \sum_{r=0}^{\infty} \bar{f}_r \bar{d}_{r+1},
\]  
(5.273)

with the added constraint that \(p_B\) is real.

We can now use the definitions of \(l_j\) in (5.267) and (5.271) to evaluate \(iq\chi\) in (5.264). Being careful with the arguments, we find that on \(B\),
\[
\chi = 2 \xi^{1/2} \chi_0^{1/2} R,
\]  
(5.274)
where
\[
\chi_0 = \exp \left[ \frac{1}{2} (a + b) \right], \quad R = \left[ \sin \left( \frac{x - a}{2} \right) \sin \left( \frac{b - x}{2} \right) \right]^{1/2},
\]
and after some algebra, we have the differential equation for \(s\) on \(B\):
\[
s' = s'_0 - 4R \Im \left[ \chi_0^{1/2} \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} f_{r}d_{j+r+1} \exp \left\{ i \left( j + \frac{1}{2} \right) x \right\} \right],
\]
with initial condition \(s(a) = s_0(a)\). To solve this, guess \(b\); we can then calculate the right hand side. Solving for \(s\), we adjust \(b\) by decreasing it if \(s\) reaches \(s_0\) for \(x < b\), and increase it if \(s\) remains > \(s_0\) for all \(x \leq b\).

**Computational approaches**

Complex analysis is all very elegant, but is probably not an efficient way to compute a time-evolving interface. A direct computational approach would be preferable, but the free boundary nature renders this problematic. Two ways of dealing with this issue have been suggested, and are discussed further in the notes.

### 5.9 Notes and references

Books describing sediment transport and its effects on river morphology include those by Allen (1985), Ahnert (1996), Knighton (1998) and Goudie (1993). Mention must also be made of Gary Parker’s e-book (Parker 2004), which describes in the form of powerpoint lectures a wealth of phenomena and theory concerning river bedforms. The classical book on aeolian dunes is that of Bagnold (1941), and a more recent classic is that of Pye and Tsor (1990). Both books have recently been reprinted, Bagnold’s by Dover in 2005, and Pye and Tsor’s by Springer in 2009.

**Linear stability**

The first theory for dune and anti-dune formation which embodied the principle of upstream stress migration was due to Kennedy (1963), as described in section 5.3. Kennedy was motivated by Benjamin’s earlier (1959) result on laminar fluid flow over small bumps, but the prescription of a fixed spatial lag is flawed. Parker (1975) suggested that the inertial effect of bedload (i.e., sediment flux relaxes to its equilibrium value over a finite length) could be a causative mechanism for the formation of anti-dunes.

St. Venant-type models were introduced by Reynolds (1965), and the failure of averaged models to locate instability led Engelund (1970) and Smith (1970) to study eddy viscosity type models in which the two-dimensional nature of the flow was paramount. Subsequent developments of the instability theory were made by Fredsoe (1974), Richards (1980) (who extended the theory to the formation of ripples), Sumer and Bakioglu (1984), Colombini (2004) and Charru and Hinch (2006).
Sediment transport

The Shields stress, and the experimental data in figure 5.7, were given in his thesis by Shields (1936). There are a number of empirical estimates for fluvial bedload transport, of which that described by Meyer-Peter and Müller (1948) (see also Einstein 1950) is a popular one, though possibly not the best. Similar relations are found for aeolian sand transport (e.g., Bagnold 1936, Pye and Tsoar 1990). Formulae describing the rate of entrainment or erosion of sediments into suspension are given by García and Parker (1991), Van Rijn (1984), and Smith and McLean (1977), for example.

Turbulent flow and eddy viscosity

The use of an eddy viscosity gives the simplest description for a turbulent flow, but as mentioned in section 5.7.4, the choice of eddy viscosity is problematic. Prandtl’s mixing length theory for a shear flow

$$\tau = \kappa^2 z^2 \left| \frac{\partial u}{\partial z} \right| \frac{\partial u}{\partial z}$$

(5.277)

correctly yields the logarithmic velocity profile, but is frame dependent, as well as having an infinite velocity at the wall. Usually (e.g., Schlichting 1979[10]) one retains the no slip condition by specifying a wall roughness, which has the effect of applying the no slip condition at a finite elevation $z = z_0$. An alternative (and preferable) method is to include the small laminar viscosity, thus replacing (5.277) by

$$\tau = \left( \varepsilon + \kappa^2 z^2 \left| \frac{\partial u}{\partial z} \right| \right) \frac{\partial u}{\partial z},$$

(5.278)

in suitably scaled variables; essentially $\varepsilon = 1/Re$. Solution of a constant shear stress shear flow satisfying $u = 0$ at $z = 0$ shows that the effective roughness concept can be applied, where (for (5.278)), we find (see question 5.11) $z_0 = \frac{\varepsilon}{2\kappa}$.

The frame indifference issue could be resolved by using Von Kármán’s version of the mixing length, which replaces (5.278) with

$$\tau = \left( \varepsilon + \frac{\kappa^2 |u_2|^3}{|u_{2z}|^2} \right) \frac{\partial u}{\partial z},$$

(5.279)

this is of course also not frame indifferent, but can easily be made so by generalising to, for example,

$$\tau = 2\eta \dot{\varepsilon}, \quad \dot{\varepsilon} = \frac{1}{2} (\nabla u + \nabla u^T),$$

(5.280)

where the effective viscosity is

$$\eta = \varepsilon + \frac{2\kappa^2 |\dot{\varepsilon}|^3}{|\nabla \cdot \dot{\varepsilon}|^2}.\)

(5.281)

[10] The Schlichting book went through many reprints, and currently exists in print in a revised edition by Schlichting and Gersten published by Springer in 2000; this new book is quite different from the earlier version, and a good deal of material in the original book has been removed.
However, (5.279) is also problematical, because it allows $\eta$ to depend on the second derivative of $u$, thus artificially raising the order of the equations. Taking the mixing length $l = \frac{\kappa}{|u_z|}$ does not work. The only other possibility along these lines might be to assume a dependence on fractional derivatives of $u_z$, although there seems little physical justification for this.

**Jackson–Hunt theory**

The classic paper describing turbulent flow over a small hill is that by Jackson and Hunt (1975). Further developments of the theory are given by Sykes (1980), Hunt et al. (1988) and (less easy to find) Weng et al. (1991). It is generally acknowledged that the Jackson–Hunt paper is very difficult to read. The theory is complicated for one thing, but the manner of presentation is not clear. Rather than present a clearly stated boundary value problem, Jackson and Hunt present solutions, model, approximations, scales and limits all mixed together. Sykes (1980) provided a more rational asymptotic treatment of the problem, and pointed out various difficulties in the Jackson–Hunt theory, but like them, Sykes avoided providing a description for the Reynolds stresses until late on. Thus, the Jackson–Hunt theory divorces the assumed basic logarithmic velocity from the assumed form for the Reynolds stresses. The version of the theory presented here, in section 5.7.2, adopts a different philosophy, that the logarithmic profile must itself be a consequence of the boundary value problem to be solved. While this may seem a sensible approach, it raises the issue of how best to prescribe the Reynolds stresses. Sykes provides a fairly sophisticated closure scheme, without an indication that the basic solution has the required logarithmic profile.

Hunt et al. (1988) provide an improved version of the theory, which we summarise here. The paper is again difficult to read. The basal shear stress $\tau$ is given by equation (3.1)$_H$ (all equation numbers with subscripts $H$ refer to Hunt et al. (1988))

$$\tau = \varepsilon^2 \rho U_0^2 (1 + \tau_d).$$

(5.282)

At the top of page 1,439, we find

$$\varepsilon = \frac{u_\ast}{U_\infty},$$

(5.283)

where $u_\ast$ is the friction velocity, and $U_\infty$ is the far-field velocity, essentially the same as our definition in (5.121), whereas $U_0$ is the velocity of the basic profile at a height $h_m$; however in equation (2.4c)$_H$ we have $\varepsilon = \frac{u_\ast}{U_0}$, so we will suppose that $U_0 \approx U_\infty$, which is also consistent with the discussion at the very bottom of page 1,438 and the top of page 1,439.

The perturbation shear stress $\tau_d$ is defined in (3.7d)$_H$, and using (3.12a,b)$_H$ its Fourier transform\(^{11}\) is given by

$$\hat{\tau}_d = -\frac{2p_0}{U^2(l)} \sigma(k)[1 + \delta(2 \ln k + 4 \gamma + 1 + i\pi)].$$

(5.284)

\(^{11}\)Defined, as I have also done here in (5.187), via $\int \ldots e^{-ikx} dx$, and denoted by an overhat.
\( p_0 \) is defined in (2.15)_H, as is \( \sigma \), as minus the Hilbert transform of the bed slope. \( U(l) \) is the (scaled, with \( U_0 \) in (2.1)_H) velocity of the undisturbed flow at a height \( l \) above the bed, and is defined (at the bottom of page 1,438) by

\[
U(l) = \frac{\varepsilon}{\kappa} \ln \left( \frac{l}{z_0} \right), \tag{5.285}
\]

where \( z_0 \) is the roughness length; (5.285) assumes \( l \ll h_m \) (as is the case), while \( l \) is defined in (3.6)_H by

\[
\ln \left( \frac{l}{z_0} \right) = 2\kappa^2 d \tag{5.286}
\]

(the depth scale of the flow \( d \) is denoted \( L \) in the Hunt paper). The small parameter \( \delta \) in (5.284) is defined on page 1,449, two lines above (3.7a)_H:

\[
\delta = \frac{1}{\ln \left( \frac{l}{z_0} \right)} \tag{5.287}
\]

It is convenient to define

\[
l = 2\kappa \varepsilon \Lambda d, \tag{5.288}
\]

and then we have

\[
\Lambda = \frac{1}{1 + \frac{\varepsilon}{\kappa} \ln 2\kappa \varepsilon \Lambda} \approx 1 - \frac{\varepsilon}{\kappa} \ln 2\kappa \varepsilon, \tag{5.289}
\]

and

\[
\delta = \frac{\varepsilon \Lambda}{\kappa}. \tag{5.290}
\]

Using these results, we find that the dimensionless Hunt formula for the basal shear stress can be written as

\[
\tau = 1 + \varepsilon B_1 + \varepsilon^2 H_2 + \ldots, \tag{5.291}
\]

where

\[
B_1 = -2H(s_x), \tag{5.292}
\]

as in (5.198), and the transform of \( H_2 \) is

\[
\hat{H}_2 = \frac{\hat{B}_1}{\kappa} \{ -2 \ln 2\kappa \varepsilon + 2 \ln k + 4\gamma + 1 + i\pi \}. \tag{5.293}
\]

We can now compare the results with the formula derived in 5.7. The formulae (5.197) and (5.291) differ in the \( O(\varepsilon^2) \) coefficient, and these are related, assuming \( -k = |k|e^{-i\pi} \) (as is required: see the comment following (5.188) and its accompanying footnote), by

\[
B_2 = H_2 + C, \tag{5.294}
\]

where the transform of \( C \) is defined in (5.200). The difference between the two versions of the theory lies in the way in which the Reynolds stress terms are treated when they occur at second order. Since it is the second order terms which provide the instability, we see that the matter of their computation is of some importance. The difference presumably arises because Jackson and Hunt do not make explicit their assumption on the Reynolds stress away from the boundary.
The Herrmann model

The principal exponent of dune modelling is Hans Herrmann, and there is also a thriving French school under the aegis of Bruno Andreotti. The basis of the Herrmann approach is in the papers by Sauermann et al. (2001), Kroy et al. (2002a) and Kroy et al. (2002b), which last is simply a more complete exposition of their earlier paper. The Herrmann model is essentially an Exner-Hunt model, that is to say that the Exner model $s_t + q_x = 0$ is combined with a Bagnold-type transport law $q = q_0(\tau)$, in which a lag is included to represent the finite acceleration of the transport, thus, essentially,

$$i\varepsilon \frac{\partial q}{\partial x} = q_0 - q,$$

(5.295)

and finally the stress is computed using the Jackson-Hunt theory. From (5.291), (5.292) and (5.293), we can write the transform of the stress perturbation, $\hat{\tau}_1 = \tau - 1$, in the form (assuming $k = |k|e^{-ix}$ when $k < 0$)

$$\hat{\tau}_1 = \varepsilon (A|k| + iBk)\hat{s},$$

(5.296)

where

$$A = 2 \left[ 1 + \frac{\varepsilon}{\kappa} \left\{ 2 \ln \left( \frac{|k|}{2\kappa \varepsilon} \right) + 4\gamma + 1 \right\} \right],$$

$$B = \frac{2\pi \varepsilon}{\kappa}.$$  

(5.297)

Kroy et al. (2002b) give the same formula (5.296) (their equation (12), the extra $\varepsilon$ arising when their formula is made dimensionless), but their definitions of $A$ and $B$ are not quite the same, although also based on the Hunt formula. The values are similar though; based on values $|k| = 1$, $\varepsilon = 0.03$, $\kappa = 0.4$ corresponding to $\frac{dz_0}{d} = 0.6 \times 10^6$, we calculate $A = 3.6$, $B = 0.47$, compared to the typical Kroy values $A \approx 4$, $B \approx 0.25$.

The linearised Herrmann model for the transforms of the perturbed variables takes the form (cf. (5.201), (5.295) and (5.296))

$$\mu ik\hat{q} = q_0^\prime \hat{\tau} - \hat{q},$$

$$\varepsilon \hat{s}_t + ik\hat{q} = 0,$$

$$\hat{\tau} = \varepsilon (A|k| + iBk)\hat{s},$$

(5.298)

where the relaxation length parameter $\mu$ is

$$\mu = \frac{l_s}{d},$$

(5.299)

and is small. With $\hat{s} \propto e^{\sigma t}$, we obtain

$$\sigma = r - ikc = \frac{-ikq_0^\prime(A|k| + iBk)}{1 + \mu ik},$$

(5.300)
and thus the growth rate is

\[ r = \frac{q_0 k^2(B - \mu A|k|)}{1 + \mu^2 k^2}, \tag{5.301} \]

and the wave speed\(^{12}\) is

\[ c = \frac{q_0 (A|k| + \mu B k^2)}{1 + \mu^2 k^2}. \tag{5.302} \]

**Fluvial versus aeolian?**

The Herrmann version of the theory is very attractive because the relaxation length causes the growth rate to become negative at large wavenumber. This is likely relevant for aeolian dunes, but less relevant for fluvial dunes, where one might expect \( \mu \) to be tiny. However, the instability relies on the parameter \( B > 0 \), and if the downslope term in (5.224) is included, then the definition of \( B \) in (5.297) is modified to

\[ B = \frac{2\pi \varepsilon}{\kappa} - \frac{\hat{\beta}}{\varepsilon}, \tag{5.303} \]

indicating \( B < 0 \) and stability. The constant eddy viscosity (Benjamin) model does not suffer this defect because then the growth rate is proportional to \( k^{4/3} \). On the other hand, we expect the Hunt theory to be more accurate.

There is thus a conundrum in how the models are designed. In aeolian bed transport, the sand grains are transported by saltation in a layer of tens of centimetres depth. It is likely to be the case that this finite thickness has a quantitative effect on the application of the Hunt theory. In addition, the rôle of the downslope term may become essentially irrelevant, if the transport is largely by saltation. Equally, the relaxation length is likely to be important. Kroy et al.’s estimate is \( l_s \sim 1-2 \text{ m} \), and thus \( \mu \sim 0.002 \). With \( B \) being relatively small, the maximal growth rate from (5.301) occurs at \( k \sim \frac{2B}{3\mu A} \), corresponding to a wavelength of 300 m, if we take \( d = 1,000 \text{ m} \), \( A = 4 \), \( B = 0.5 \), \( \mu = 0.002 \).

It is not so obvious that the same will be true in fluvial transport. The thickness of the bedload layer is only a few grain diameters, and the relaxation length is likely to be very small. The downslope component of the effective shear stress may be important, and as we have seen, this also provides a stabilising (diffusive) effect. In this case, it is difficult to see how the Hunt model can produce instability.

**Separation**

The principal difficulty in applying the Jackson–Hunt theory (or indeed any theory) to dune formation lies in the tacit assumption that the flow is attached, and this is almost never the case in practice. Measurements of separated flow have been made by Vosper et al. (2002); numerical computations indicating separation have been made

\(^{12}\)The wave speed is \(-\text{Im} \sigma/k\) here because the Fourier transform is defined with \( e^{-ikx} \).
by Parsons et al. (2004), and attempts to model similar flows have been made by O’Malley et al. (1991), and also Cocks (2005), whose work on a complex variable method is described in section 5.8.1. However, the complex variable approach is unwieldy, and in any case not suitable for three-dimensional calculations.

The approach used by Herrmann and his co-workers is to get around this in a plausible but heuristic way. When the lee side slope exceeds $14^\circ$, then separation occurs, and they carry on the calculation by fitting a cubic function as the separation bubble roof. Since a cubic is defined by four parameters, but also the point of reattachment is unknown, this allows Kroy et al. (2002b) to specify five conditions; these are continuity of interface and its slope at the end points, together with a specification that the maximum (negative) slope of the bubble roof is $14^\circ$ (their equation (27). However, towards the beginning of the same paragraph, they also say that they require the curvature of the bed to be continuous; indeed, this ensures that the basal stress is continuous at the detachment point, and thus that separation occurs when $\tau = 0$, since the shear stress is zero in the bubble, but it is not clear whether their prescription satisfies this condition.

Insofar as one wants to solve a separation problem in which the shear stress is zero at the bubble roof, there are two apparent problems with the Herrmann approach. The first is that the calculation of the shear stress via (5.291), for example, involves the assumption of a no slip condition, as opposed to a no stress condition. Kroy et al. recognise this (after their equation (26)), but think that ‘the corresponding errors are expected to be small’, although why this should be so is not clear. One might in fact expect the errors to be large. The second problem is that if the bubble roof $s$ is chosen in a prescribed way, there is no particular reason to suppose that the shear stress thus calculated will actually equal zero.

Despite these misgivings, the utilisation of this model gives strikingly interesting results. Schwämmle and Herrmann (2004) studied transverse dunes, Parteli et al. (2007) studied barchan dunes, and Parteli et al. (2009) studied seif dunes. Durán and Herrmann (2006) studied the transition from barchans to parabolic dunes under the effect of vegetation. The computational results which they show are impressive, perhaps suggesting that the details of the model are not that important.

More recently, Fowler et al. (2011) have adopted a different strategy. They use a constant eddy viscosity approach, which leads to the Exner equation

$$\frac{\partial s}{\partial t} + \frac{\partial q}{\partial x} = 0,$$

with $q = q(\tau_e)$, $\tau_e$ being an effective basal stress defined by

$$\tau_e = \tau - \beta s_x, \quad \tau = 1 - s + K \ast s_x$$

(cf. (5.103)), and we allow a stabilising down slope coefficient $\beta$. Numerical solutions of this equation show that $\tau$ reaches zero, signalling the onset of separation. Thereafter, the Exner equation becomes redundant in the separation bubble, and is replaced by $\tau = 0$. Providing we assume the same formula for the stress applies when there is separation, a convenient mathematical way of formulating the problem in this
Figure 5.16: Snapshot of the travelling dune system of figure 5.12 at time $t = 2$, found by solving (5.306) and (5.307), using $q = [\tau]^{3/2}$ and where $D = 4.3$ is constant, and $M$ is given by (5.309), with $\Lambda_a = 400$ and $\Lambda_s = 20$. The upper curve is $s$, and, where distinct, the lower is the sand surface $b$. Figure courtesy of Mark McGuinness.

case is to separately compute the sand bed $z = b(x, t)$ together with the air flow base $z = s(x, t)$ (i.e., $s$ is the sand surface except in the separation bubble, when it is the roof of the bubble). We then solve the pair of equations

\[\begin{align*}
s_t + q_x &= M, \\
b_t + q_x &= 0,
\end{align*}\]  

(5.306)

with $q$ given as a function of $\tau_e$, and $M$ to be chosen. For small $s$, we can approximate $q$ by

\[q \approx q(\tau) - D s_x,\]

(5.307)

where the diffusion coefficient $D$ is

\[D = \beta q'(\tau),\]

(5.308)

and this is more convenient for numerical purposes.

The choice of $M$ is motivated by the fact that we should have $M = 0$ when $s = b$ and $\tau > 0$, but $M$ is indeterminate when $s > b$ and $\tau = 0$. A suitable computational choice is to define

\[M = \begin{cases} -\Lambda_a (s - b) & \text{if } \tau > 0, \\
-\Lambda_s \tau & \text{if } \tau < 0, \end{cases}\]  

(5.309)

where the values of $\Lambda_i$ are chosen to be large. Since $(s - b)_t = -\Lambda_a (s - b)$ when $\tau > 0$, this forces the air flow to remain attached to the sand surface, while if $\tau$ starts to become negative, ‘fake’ sand is artificially produced to inflate $s$ so as to keep $\tau \approx 0$.  

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(We will find a similar strategy to this bears fruit when modelling drumlin formation in chapter MLJE.)

Figure 5.16 shows the result of a computation with this model, corresponding to the evolution from an initial disturbance, as shown in figure 5.12. In this figure, we have taken the diffusion coefficient to be a constant, which aids numerical computation. However, this choice allows the stationary sand inside the air bubble to diffuse. More realistically, since $D \to 0$ when $\tau \to 0$, the bubble sand will steepen to form a shock at the lee of the dune, but this itself does not occur in practice because a gradient steeper than about $34^\circ$ cannot be obtained. We can model the resulting slip face by allowing the diffusion coefficient to increase without bound as $-s_x$ approaches the critical slope $S_c = \tan 34^\circ \approx 0.67$, for example by allowing $\beta \to \infty$ as $-s_x \to S_c$. This is awkward to arrange, and largely cosmetic, so long as the diffusing sand in the air bubble does not reach the downstream end of the bubble.

The shapes of the bubbles are also not very realistic, but this may be due to the incorrect calculation of the shear stress in the presence of separation. This is the second difficulty, which no model has yet addressed: the issue of prescribing the shear stress when there is separation. The simplest situation is the constant viscosity model, in which there will now be ordinary Blasius boundary layers which join the attached flow to an outer flow in which there is slip past the boundary. The degree of slip must be calculated as a consistency condition with the boundary layer solution. This has yet to be done, but the structure of the resulting dune theory is likely to be very different.

**Exercises**

5.1 Just as the straightforward St. Venant model is unable to predict the occurrence of transverse dunes, it is also apparently unable to produce lateral bars; at least, this is suggested by the following example.

Show that a two-dimensional form of the St. Venant equations describing flow in a stream of constant width, which allows for downslope sediment transport, can be written in the dimensionless form

$$s_t + \nabla \cdot q = 0,$$

$$q = \frac{q(\tau_e)}{\tau_e} \tau_e,$$

$$\tau_e = |u| - \beta \nabla s,$$

$$\varepsilon h_t + \nabla \cdot (h u) = 0,$$

$$F^2 [\varepsilon u_t + (u \nabla)u] = -\nabla \eta + \delta \left( i - \frac{|u| u}{h} \right),$$

$$h = \eta - s.$$

Assume that $\beta \sim O(1), F \sim O(1), \delta \ll 1,$ and $\varepsilon \ll 1$. Suppose also that the cross stream width $y \sim \nu \ll 1$. Show that it is appropriate to rescale the
transverse velocity \( v \) (i.e., \( \mathbf{u} = (u, v) \)) as \( v \sim \nu \), and then also \( s \sim \nu^2 \) and \( t \sim \nu^2 \). Assuming that \( \varepsilon \ll \nu^2 \) and that \( q = \tau_x^{3/2} \), show that a consistent approximate rescaled model is

\[
\frac{\partial s}{\partial t} + \frac{\partial (u^3)}{\partial x} + \frac{\partial (u^2 v)}{\partial y} = \beta \left( u \frac{\partial s}{\partial y} \right),
\]

\[
\frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0,
\]

\[
F^2 uu_x + vu_y + \eta_x = 0,
\]

and that \( \eta \approx \eta(x, t), h \approx \eta \). Deduce that \( s \) satisfies the equation

\[
\frac{\partial s}{\partial t} = \left( \frac{2u}{F^2} + \frac{u^3}{h} \right) h_x + \beta \frac{\partial}{\partial y} \left( u \frac{\partial s}{\partial y} \right).
\]

For small perturbations to the uniform state \( h = u = 1, s = 0 \), show that \( F^2 uu_x + hu_x \approx 0 \) in a linearised approximation, and deduce that \( u \approx u(x, t) \).

Show that then \( s \) relaxes to a steady state, and by considering suitable boundary conditions at the stream margins, show that in fact \( h_x = 0 \), and hence the uniform state is stable.

Now suppose that the stream is not supposed narrow, so that the rescaling with \( \nu \) is not done. Show that for sufficiently small spanwise perturbations such that we can still take \( |u| \approx u, \tau_c \approx u^2 \) and \( q \approx u^3 \), the model may be reduced to

\[
\frac{\partial s}{\partial t} + \frac{\partial (u^3)}{\partial x} + \frac{\partial (u^2 v)}{\partial y} = \beta \left( \frac{\partial}{\partial x} \left( u \frac{\partial s}{\partial x} \right) + \frac{\partial}{\partial y} \left( u \frac{\partial s}{\partial y} \right) \right),
\]

\[
\frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0,
\]

\[
F^2 uu_x + vu_y + h_x + s_x = 0,
\]

\[
F^2 uu_x + vu_y + h_y + s_y = 0.
\]

By linearising about the uniform state, show that perturbations proportional to (the real part of) \( \exp[\sigma t + ik_1 x + ik_2 y] \) have a growth rate determined by

\[
\sigma = -\beta k^2 - \frac{ik_1(3k_1^2 + k_2^2)}{k_2^2 + (1 - F^2)k_1^2},
\]

where \( k^2 = k_1^2 + k_2^2 \). Deduce that perturbations take the form of decaying travelling waves, and comment on the direction of propagation for purely longitudinal and purely transverse waveforms.

5.2 Write down the Exner equation for bedload transport, and show how it can be used to study the onset of bedform instability, assuming a suitable bedload transport law. Show that in conditions of slow flow, the resultant equation for
the bed profile \( s(x, t) \) is a first order hyperbolic equation, and deduce that the profile is neutrally stable. Show also that bed waves will form shocks which propagate downstream.

Now suppose that the bedload transport \( q_b(x, t) \) is a function of the basal stress \( \tau \) evaluated at \( x - \delta \). Show that instability can occur if \( \delta < 0 \), i.e., the stress leads the bed profile.

Can you think of a physical reason why such a lead should occur?
Do you think such a model would be a good nonlinear model?

5.3 The Kennedy model for dune growth leads to the dispersion relation

\[
\sigma [(\sigma + ikU)^2 + gk \tanh kh] + (\sigma + ikU)kq \cdot e^{-ik\delta}[(\sigma + ikU)^2 \tanh kh + g] = 0,
\]

where \( \sigma \) is the growth rate and \( k \) is the wave number.

Show that if \( q' \) is small, then \( \sigma \approx -ikc_\pm \), where

\[
c_\pm = U \pm \sqrt{\frac{g}{k} \tanh kh}.
\]

Use this result to show, by considering a correction to this approximate value, that

\[
\text{Re} \sigma \approx -\frac{gkq'}{2c_\pm} \sin k\delta \text{sech}^2 kh,
\]

and deduce that forwards travelling waves are (weakly) unstable if \( \sin k\delta < 0 \).

5.4 In Reynolds' model of dune formation, the bed elevation is \( s \), the surface elevation is \( \eta \), the water speed is \( u \), and the sediment flux is \( q \), and these are related by the equations

\[
\eta = 1 + \frac{1}{2}(1 - u^2),
\]

\[
s = \eta - \frac{1}{u},
\]

\[
u = q^{1/3},
\]

and \( q \) satisfies the Exner equation in the form

\[
q_t + v(q)q_x = 0,
\]

where the wave speed \( v(q) = \frac{dq}{ds} \).

Show that

\[
v(q) = \frac{3q^{4/3}}{1 - F^2q},
\]

and deduce that for perturbations to the steady state \( q = 1 \), waves propagate forwards if \( F < 1 \) and backwards if \( F > 1 \). By consideration of \( v'(q) \), show also that for \( F < 1 \), waves will form forward-facing shocks in \( q \) and thus also \( s \).
while if $1 < F < 2$, waves form backward-facing shocks as elevations in $s$ (and $\eta$).

What happens in this model if $F > 2$? What do you think would happen in practice?

5.5 In a model of dune formation, the sediment concentration $c$ and bed height $s$ are modelled by the equations

$$\frac{\partial}{\partial t}(hc) + \frac{\partial}{\partial x}(hc u) = \rho_s(v_E - v_D),$$

$$(1 - n) \frac{\partial s}{\partial t} = -(v_E - v_D),$$

where $h$ is fluid depth, $u$ is mean fluid velocity, $\rho_s$ is sediment density, $n$ is bed porosity, and $v_E$ and $v_D$ are erosion and deposition rates. Parker (1978) suggests the following expressions for the erosion and deposition rates in a stream:

$$v_E = \frac{\beta u_s^3}{v_s^2}, \quad v_D = \frac{\gamma u_s^2 c}{\rho_s u_s},$$

where $c$ is the sediment concentration (mass per unit volume), $v_s$ is the settling velocity, $u_s$ is the friction velocity $(\tau/\rho_w)^{1/2}$, and $\beta$ and $\gamma$ are constants ($\approx 0.007$ and 13, respectively).

Consider the two cases where (i) the surface $\eta = h + s$ is flat, and $\eta = \eta_0$ is constant; and (ii) where the surface is determined by a local force balance, thus

$$\tau = \rho_w gh(S - \eta_x),$$

where $\rho_w$ is water density, $g$ is gravity, and $S$ is bed slope.

Assuming $\tau = \rho_w u^2$ and $uh = q$ is constant, find appropriate scales for $x$, $t$ and $c$ in cases (i) and (ii) if $h, \eta, s \sim \eta_0$ and $q$ is the fluid flux per unit width. Hence derive the dimensionless model for slow flow

$$\varepsilon \frac{\partial}{\partial t}(hc) + \frac{\partial c}{\partial x} = \frac{1}{h^3} - ch = -\frac{\partial s}{\partial t},$$

where

$$\varepsilon = \frac{c_0}{\rho_s(1 - n)}.$$ 

Show that in case (i), $h = 1 - s$, while in case (ii),

$$\frac{1}{h^3} = 1 - \Lambda \eta_x,$$

and define the parameter $\Lambda$ in the second case. By analysing the stability of the basic state $h = c = \eta = 1$, show that, for $\varepsilon$ small, the steady state is stable. Show that in case (i), waves propagate downstream, but in case (ii), they can propagate upstream if $\Lambda$ is small enough.

More generally, derive a stability criterion in case (i) (when $\varepsilon$ is small) if $v_E = E(h)$, $v_D = cV(h)$. How is the result affected if $\varepsilon$ is not small?
5.6 A simple model of bed erosion based on the St. Venant equations can be written in dimensionless form as
\[ \varepsilon h_t + (u h)_x = 0, \]
\[ F^2(\varepsilon u_t + uu_x) = -\eta_x + \delta \left( 1 - \frac{u^2}{h} \right), \]
\[ h(\varepsilon c_t + uc_x) = E(u) - c = -s_t, \]
where \( h = \eta - s \). Explain a plausible basis for the derivation of this model. By considering the stability of the steady state \( u = h = c = 1 \) on a time scale \( t \) of \( O(1) \), and assuming that \( \delta \ll 1, \varepsilon \ll 1 \), show that instability can occur depending on the size of \( E'(1) \). Show also that \( \eta \) and \( s \) are out of phase if \( F < 1 \), and in phase if \( F > 1 \); interpret this in terms of dune and anti-dune formation.

5.7 The Exner equation for bed evolution is written in the form
\[ (1 - n)s_t + q_x = 0, \]
and the bedload transport is given by
\[ q_x = K [q_0(\tau) - q], \]
where \( \tau \) is the bed shear stress. Explain in physical terms why such an equation should be appropriate to describe bedload transport.

Suppose it is assumed that the depth of the flow \( h \) is constant. Show that if the bed stress is \( \tau = \rho u^2 \), then the momentum equation of St. Venant can be written in the approximate form
\[ \frac{h}{2f} \tau_x + \tau = \rho gh(S - s_x). \]
Show how to non-dimensionalise these equations to obtain the set
\[ s_t + q_x = 0, \]
\[ \delta q_x = q_0(\tau) - q, \]
\[ \tau_x + \tau = 1 - s_x, \]
and identify the parameter \( \delta \).

Write down a suitable steady state solution, and show that if \( q_0(\tau) \) is a monotonically increasing function of \( \tau \), then the steady state is linearly unstable if \( K > 0 \). Show also that the corresponding waves move upstream. Show that the growth rate remains positive as the wavenumber \( k \to \infty \). [This is an indication of ill-posedness.]

For what values of the Froude number might the assumption of constant depth be valid?
5.8 Suppose that
\[ s = s(u) = \frac{1}{2} F^2 (1 - u^2) + 1 - \frac{1}{u}, \]
and that
\[ s'(u) \frac{\partial u}{\partial t} = c D^*(u) - E^*(u) = - \frac{\partial c}{\partial x}. \]
Assume \( D^* = 1 \) and \( E^* = u^3 \). Simplify the equations to the form
\[ \frac{\partial u}{\partial t} = f(u, c), \quad \frac{\partial c}{\partial x} = g(u, c), \]
giving expressions for \( f \) and \( g \).

Suppose that \( c = 1 \) at \( x = 0 \) and \( u = u_0(x) \) at \( t = 0 \), and that \( u_0(\infty) = 1 \). Derive an ordinary differential equation for \( U(t) = u(0, t) \) in the form \( \frac{dU}{dt} = h(U) \), and by consideration of the graphical form of \( h(U) \) in the two cases \( F < 1 \) and \( F > 1 \), determine the behaviour of \( U \) for \( t > 0 \), explaining in particular how it depends on \( U(0) \).

Why is it inadvisable to prescribe \( c \to 1 \) as \( x \to \infty \) instead of the boundary condition at \( x = 0 \)?

5.9 Show that, if \( \nu > 0 \),
\[ \int_0^\infty \theta^{\nu-1} e^{i\theta} d\theta = \Gamma(\nu) \exp \left( \frac{i\pi \nu}{2} \right). \]

Hence, if
\[ K(\eta) = \begin{cases} \eta^{\nu-1}, & \eta > 0, \\ 0, & \eta < 0, \end{cases} \]
where \( 0 < \nu < 1 \), show that the Fourier transform, defined here as
\[ \hat{K}(k) = \int_{-\infty}^{\infty} K(\eta) e^{-ik\eta} d\eta \]
is given by
\[ \hat{K}(k) = \frac{\Gamma(\nu) \exp \left( -\frac{i\pi \nu \text{sgn} k}{2} \right)}{2\pi |k|^\nu} \]
for real values of \( k \).

Now suppose that \( \phi \) satisfies the evolution equation
\[ \phi_t + \frac{\partial}{\partial \xi} \left[ \frac{1}{2} \phi^2 + \alpha K * \phi - \phi \right] = 0, \]
where \( f \ast g \) denotes the Fourier convolution of \( f \) and \( g \) and \( \alpha \) is small. Show that the steady state \( \phi = 0 \) is linearly unstable, and find the wave number of the maximum growth rate.
When this equation is solved numerically, coarsening occurs, with the wavelength of the bedforms increasing with time. Show that if $\xi \sim L \gg 1$, the equation can be approximated by Burgers’ equation with small diffusivity. Hence explain the way in which coarsening occurs.

5.10 Suppose that
\[ \psi(t) = \sum_{r=0}^{\infty} f_r t^r = (1 - \xi_a t)^{-1/2}(1 - \xi_b t)^{-1/2}, \]
and it is desired to calculate the coefficients $f_r$ numerically. By consideration of the power series for $\psi^2$ (or otherwise!) show that an iterative recipe for $f_n$ is
\[ f_0 = 1, \]
\[ 2f_n = \left( \frac{\xi_{a+1} - \xi_{b+1}}{\xi_a - \xi_b} \right) - \sum_{s=1}^{n-1} f_s f_{n-s}. \]

5.11 A shear flow is described by the dimensional equation
\[ \tau = \rho \left( \nu + \kappa^2 z^2 \left| \frac{\partial u}{\partial z} \right| \right) \frac{\partial u}{\partial z}, \]
where $\nu$ is the kinematic viscosity. Show that a suitable dimensionless form is
\[ \tau = \left( \varepsilon + \kappa \varepsilon z^2 \left| \frac{\partial u}{\partial z} \right| \right) \frac{\partial u}{\partial z}, \]
where $\varepsilon = \frac{1}{Re}$, and $Re$ is the Reynolds number. Use the method of strained coordinates\(^{13}\) (i.e., write $z = s + \varepsilon z_1(s) + \ldots$, $u \sim u_0(s) + \varepsilon u_1(s) + \ldots$) to show that
\[ u \approx \frac{1}{\kappa} \ln \left( \frac{z + s_0}{s_0} \right), \]
where $s_0 = \frac{\varepsilon}{2\kappa}$ in order to suppress higher singularities in $u_1$.

\(^{13}\)See, for example, Van Dyke (1975).