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The mean field of weakly coupled oscillators exhibits non-smooth phase noise

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Abstract. – The magnitude of the complex-valued mean field of N coupled oscillators is known to act as an order parameter for the synchronization transition; in this letter it is shown that the phase speed (rate of change of the phase angle) of the mean field is a non-smooth random process, as demonstrated by the existence of a $1/f$ spectrum across a range of frequencies. An exact expression for the phase speed correlation function is derived in the limit of vanishing coupling between oscillators; numerical results show the $1/f$ scaling can also persist beyond the synchronization threshold.

Kuramoto's [1] model of coupled limit-cycle oscillators has been widely influential, with applications to synchronizing systems in a diversity of fields from biology [2,3], to Josephson junctions [4], and semiconductor laser arrays [5,6]. Strogatz [7] gives a pedagogical review of Kuramoto's mean-field approach and its historical context; the theoretical predictions of synchronization have recently been confirmed experimentally using $N = 64$ chemical oscillators [8]. Synchronization of the ensemble is generally described in terms of the amplitude of the mean field of the oscillators; in this letter we highlight the phase speed of the mean field, and show that it exhibits $1/f$ fluctuations at high frequencies.

The term "$1/f$ noise" is applied to random processes in time whose power spectra scale with frequency f as $f^{-\alpha}$ for $\alpha \approx 1$ over a broad range of frequencies. Such processes are remarkably common in many applications in physics, engineering and applied sciences, see for example the website [9] and review article [10]. Electronic engineers are especially interested in $1/f$ noise and modelling thereof, as it is an important and poorly-understood component of phase noise in oscillator circuits [11]. Many models of $1/f$ noise have been proposed, including an exactly-solvable model in discrete time [12]; in this paper we highlight the existence of $1/f$ scaling in the continuous-time phase of the mean field of an ensemble of oscillators.

In the Kuramoto model the phases $\theta_j(t)$ of N coupled oscillators are described by the equations

$$\dot{\theta}_j = \omega_j + \frac{K}{N}\sum_{k=1}^{N} \sin\left(\theta_k - \theta_j\right), \qquad j = 1, \ldots, N, \tag{1}$$

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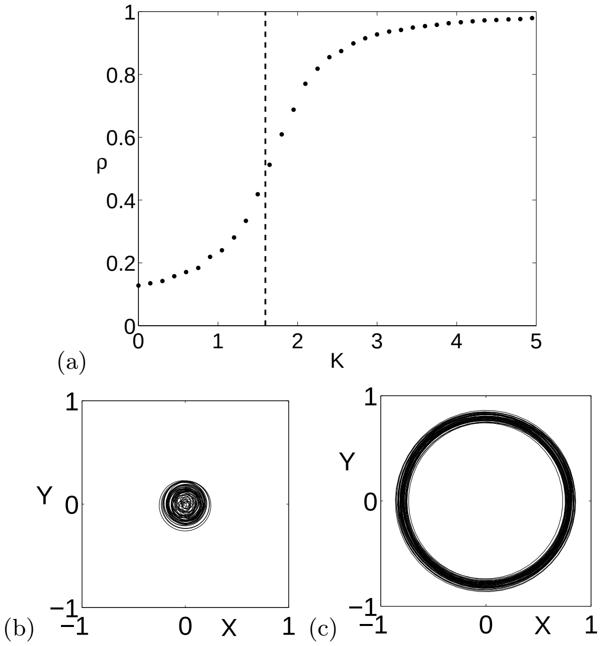

Fig. 1 – (a) Mean-field amplitude ρ as a function of coupling strength K, averaged over 100 realizations, for $N = 50$ oscillators. The critical coupling strength K_c is shown with the dashed line. (b) An example of a mean-field trajectory in the (X, Y) plane, with $K = 0.3$ and $\Omega = 2\pi$. (c) As (b), but with $K = 1.9$.

where the dot denotes time derivative. The native frequencies ω_j are chosen from a distribution $g(\omega)$ with mean Ω; note that Ω may be set to zero without loss of generality by transforming to a rotating reference frame: $\theta_j \to \theta_j + \Omega t$, $\omega_j \to \omega_j - \Omega$. The coupling parameter K determines the strength of interactions between individual oscillators. In the uncoupled ($K = 0$) case the oscillators are independent, but for sufficiently large K the individual oscillators synchronize to a common frequency. The transition to synchronization is usually expressed in terms of the complex mean field of the oscillators, defined as

$$\rho\, e^{i\Theta} = X + iY = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}. \tag{2}$$

The magnitude ρ of the mean field is commonly used to indicate the onset of synchronization. Kuramoto [1] showed that (in the $N \to \infty$ limit) ρ vanishes for $K < K_c$, where K_c is the critical coupling strength given by $K_c = 2/\pi g(\Omega)$. However for $K > K_c$ the mean-field magnitude grows as $\sqrt{K - K_c}$ near the critical point, indicating a transition to synchronization (fig. 1(a)). Trajectories in the (X, Y)-plane are confined near the origin for $K < K_c$ (fig. 1(b)), but trace out a limit cycle of radius ρ for $K > K_c$ (fig. 1(c)). In the $N \to \infty$ limit the angular speed of motion about the limit cycle is $\dot{\Theta} = \Omega$.

The effect of finite-N fluctuations upon these $N \to \infty$ theoretical results has been considered in a number of papers [13, 14]. The mean-field fluctuations scale as $N^{-1/2}$, and so the synchronization transition at $K = K_c$ is somewhat "smeared" for finite N. Figure 1(a), for example, shows the average of ρ over 100 realizations at various coupling strengths for $N = 50$ oscillators. The distribution of native frequencies is a Gaussian of mean Ω, with the variance normalized to unity, so the $N \to \infty$ critical point is $K_c \approx 1.60$. The effect of the finite-N fluctuations is clear; we note the similarity with the experimental results of figure 2B of Kiss *et al.* [8], where $N = 64$ oscillators were used. The finite-N case has also recently been shown to exhibit complex high-dimensional chaotic behavior [15], while understanding of the desynchronization mechanism has been advanced by the study of bifurcations in the $N = 3$ and $N = 5$ cases [16].

The aim of this work is to highlight the effect of finite-N fluctuations upon the mean-field phase speed $v(t) = \dot{\Theta}$. In particular we show that the stationary process $v(t)$ has a spectrum scaling as $1/f$ at high frequencies. In numerical experiments the time-dependent mean-field phase speed may be calculated as

$$v(t) = \dot{\Theta} = \frac{\mathrm{d}}{\mathrm{d}t} \tan^{-1} \frac{Y}{X}$$

$$= \frac{X\dot{Y} - Y\dot{X}}{X^2 + Y^2} \tag{3}$$

$$= \frac{\sum_{k=1}^{N} \sum_{j=1}^{N} \dot{\theta}_j \cos(\theta_k - \theta_j)}{\sum_{k=1}^{N} \sum_{j=1}^{N} \cos(\theta_k - \theta_j)}. \tag{4}$$

We calculate the spectrum of v by numerically solving the coupled equations (1) for $N = 50$ oscillators at various coupling parameters K, taking the random initial phases $\theta_j(0)$ as uniformly distributed. We choose a Gaussian distribution $g(\omega)$ of native frequencies with mean zero: $\Omega = 0$. The variance ω_0 of the distribution (if it exists) defines a characteristic dynamical timescale as ω_0^{-1}; all results are reported in non-dimensional time $\tilde{t} = \omega_0 t$, thus the non-dimensional variance of $g(\omega)$ is unity. After neglecting transients, in each realization a time series $v(t)$ of 2^{14} samples at intervals $\Delta t = 3.125 \times 10^{-3}$ is fast-Fourier-transformed; the average spectrum over 1000 realizations is shown in fig. 2. A clear $1/f$ range in the spectrum is apparent for coupling strengths from sub-threshold levels up to the synchronization threshold at $K_c \approx 1.60$, and even beyond (see, for example, the $K = 1.9$ spectrum corresponding to mean-field trajectories such as that in fig. 1(c)). Only when $K > 2$ do all vestiges of $1/f$ scaling disappear; this corresponds to phase-locking of all oscillators to a common frequency, with negligible fluctuations. Moreover, the $1/f$ spectrum is present even in the $K = 0$ limit of uncoupled oscillators. This suggests that the $1/f$ spectrum of the mean phase speed is independent of the synchronization mechanism.

The $1/f$ spectrum may be explained intuitively by noting that for low coupling strengths the mean-field trajectory $X + iY$ sometimes moves very close to the origin (fig. 1(b)), and so the time series of the phase speed (3) shows large intermittent spikes. As the coupling strength increases, the mean field spends less time near the origin (fig. 1(c)), and so there is less of the $1/f$ contribution to the spectrum. Note that Θ is undefined if X and Y are both exactly equal to zero, but as long as either X or Y is non-zero (which occurs almost surely, *i.e.*, with probability 1) then the phase angle and the phase speed are both well defined.

The uncoupled ($K = 0$) limit may be examined analytically. In this case the mean-field components $X(t) = \frac{1}{N}\sum_{i=j}^{N} \cos(\omega_j t + \theta_j(0))$ and $Y(t) = \frac{1}{N}\sum_{j=1}^{N} \sin(\omega_j t + \theta_j(0))$ are independent and (by the central limit theorem for large N) Gaussian random functions. The autocorrelation function of X and Y defined by

$$\frac{\langle X(t)X(t+\tau)\rangle}{\langle X(t)^2\rangle} = \frac{\langle Y(t)Y(t+\tau)\rangle}{\langle Y(t)^2\rangle} = R(\tau) \tag{5}$$

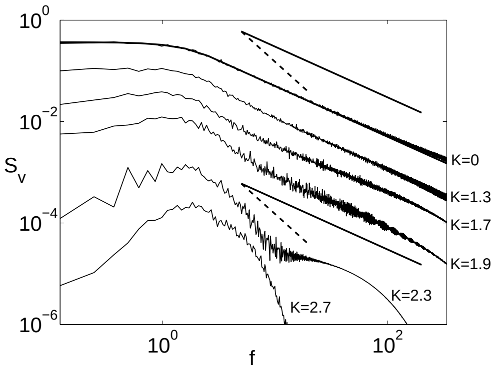

Fig. 2 – Average (over 1000 realizations) of the phase noise spectrum at various coupling strengths for Gaussian $g(\omega)$ and $N = 50$. From the top spectrum downwards, the couplings are: $K = 0$, 1.3, 1.7, 1.9, 2.3, 2.7; note the synchronization transition strength is $K_c \approx 1.60$. Solid lines correspond to $1/f$ slopes; dashed lines show $1/f^2$ for contrast. The spectrum predicted from the exact result (10) is shown as a solid curve; it is almost indistinguishable from the numerical result for $K = 0$.

is related directly to their power spectrum $S(f)$ as

$$S(f) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{if\tau} R(\tau)\,\mathrm{d}\tau = g(f), \tag{6}$$

where g is the distribution function of the native frequencies of the oscillators. It is known [17, 18] that when $X(t)$ and $Y(t)$ are independent zero-mean Gaussian processes with autocorrelation function $R(\tau)$ and finite second spectral moment, *i.e.*, $\int_{-\infty}^{\infty} f^2 S(f)\,\mathrm{d}f < \infty$, then the correlation function $R_v(\tau)$ of the phase speed $v(t)$ defined in (3) is given by

$$R_v(\tau) \equiv \langle v(t)v(t+\tau)\rangle = \frac{1}{2R(\tau)^2} \left\{R(\tau)R''(\tau) - R'(\tau)^2\right\} \log\left[1 - R(\tau)^2\right]. \tag{7}$$

Provided that the autocorrelation function $R(\tau)$ of X and Y is sufficiently smooth that $R''(0)$ exists, a small-τ expansion of the phase speed correlation $R_v(\tau)$ yields the asymptotic form

$$R_v(\tau) \sim -\frac{1}{2\tau_0^2} \log\left(\frac{\tau^2}{\tau_0^2}\right) \qquad \text{as } \tau \to 0. \tag{8}$$

Here the timescale τ_0 is defined by the initial radius of curvature of $R(t)$: $\tau_0 \equiv [-R''(0)]^{-1/2}$. The phase speed spectrum $S_v(f)$ is the Fourier transform of $R_v(\tau)$, as in eq. (6). The large frequency asymptotics of the spectrum follow from the small-τ expansion (8) of the correlation function using the method of steepest descents [19]:

$$S_v(f) \sim \frac{1}{2\tau_0^2}\frac{1}{f} \qquad \text{as } f \to \infty. \tag{9}$$

In fact for the Gaussian distribution $g(\omega)$ of native frequencies used in our numerical simulations, it is easy to see that the autocorrelation function of the mean field components X and Y is $R(\tau) = \exp\left[-\tau^2/2\right]$, and in this case eq. (7) yields the phase speed correlation function

$$R_v(\tau) = -\frac{1}{2} \log\left[1 - e^{-\tau^2}\right]. \tag{10}$$

The predicted spectrum for the uncoupled case is determined by a numerical integration of the correlation function (10), and is almost indistinguishable from the numerical results, as shown by the top spectrum in fig. 2.

The condition on the finiteness of $R''(0)$, or equivalently of $\int_{-\infty}^{\infty} f^2 S(f)\,\mathrm{d}f$, imposed in the above analysis requires the distribution function $g(\omega)$ of native frequencies to have finite variance. While this is the case for the Gaussian $g(\omega)$ used above, the condition is violated if $g(\omega)$ is a Lorentzian (Cauchy) distribution. An exact solution for the $N \to \infty$ order parameter ρ was obtained by Kuramoto [1] assuming a Lorentzian distribution of frequencies and so this distribution is popular in the coupled-oscillators literature. However, our numerical results indicate that a $1/f$ region as seen in fig. 2 for Gaussian $g(\omega)$ does not appear when $g(\omega)$ is Lorentzian, indicating that the finiteness of the variance of g is a necessary condition.

The $1/f$ tail of the spectra shown in fig. 2 has a lower-frequency cutoff at a "shoulder" frequency of order one. Our non-dimensionalization of time by the standard deviation ω_0 of the native frequencies implies that the dimensional shoulder frequency is approximately ω_0. Thus a narrower distribution $g(\omega)$ of native frequencies about the mean (*i.e.*, lower ω_0) leads to a lower frequency shoulder in the $1/f$ spectrum. Indeed as the distribution $g(\omega)$ limits to a delta-function ($\omega_0 \to 0$), the $1/f$ spectrum extends to arbitrarily low frequencies. In other words, less heterogeneity in the native frequencies (lower variance) leads to a lower cutoff for the $1/f$ range in the phase speed spectrum; thus this mechanism can produce low-frequency as well as high-frequency $1/f$ noise.

It is important to stress that the spectra shown in fig. 2 are those of the *mean-field* phase speed, and their properties, including the $1/f$ scaling, are due to the collective effects of the N oscillators. By contrast, each individual oscillator, for instance the j-th, has its own phase speed $\dot{\theta}_j$, which does not in general exhibit a $1/f$ spectrum. Indeed this is clear in the $K = 0$ limit, as the phase speed of the individual oscillator is the constant ω_j (giving a delta-spike individual spectrum), whereas the mean-field phase speed $\dot{\Theta}$ has the rich spectrum seen in fig. 2. Nevertheless, it is interesting to consider situations in which $1/f$ scaling might be observed in the phase speed spectra of individual oscillators. One such scenario arises if each oscillator has its own coupling strength K_j, so that eq. (1) is modified to

$$\dot{\theta}_j = \omega_j + K_j \rho \sin(\Theta - \theta_j), \qquad j = 1, \ldots, N, \tag{11}$$

where the dependence on the mean field (2) has also been made explicit. Suppose now that the coupling strength of the first $N-1$ oscillators is uniform and subthreshold ($K_j < K_c$ for $j = 1, \ldots, N-1$), but that the N-th oscillator is very strongly coupled to the mean field, *i.e.*, $K_N \gg K_c$. With the second term on the right-hand side of eq. (11) being dominant for $j = N$, the N-th oscillator synchronizes with the mean field on a fast timescale, and so the spectrum of $\dot{\theta}_N$ is expected to display a $1/f$ range. Figure 3 shows the results of numerical simulations of $N = 50$ oscillators with $K_1 = K_2 = \ldots = K_{N-1} = 0.3$, and for various large values of K_N. When the N-th oscillator is strongly coupled to the mean field, its phase speed develops a $1/f$ spectrum over a frequency range which grows with the coupling strength.

All our numerical experiments have used $N = 50$ oscillators, so the mean-field components exhibit quite large finite-size fluctuations about their $N \to \infty$ limits. We use Daido's [13] numerical and theoretical results for the mean-field components $X(t)$ and $Y(t)$ to predict the phase speed spectra for large N. While the $K = 0$ case may be examined analytically as above, the most interesting result of this paper is the persistence of the $1/f$ spectrum as the coupling in increased to the synchronization threshold (see fig. 2). We note that Popovych *et al.* [15] have recently demonstrated the existence of phase chaos (positive Lyapunov exponents) in this $K < K_c$ parameter regime. Following Daido, we write the mean-field components for this

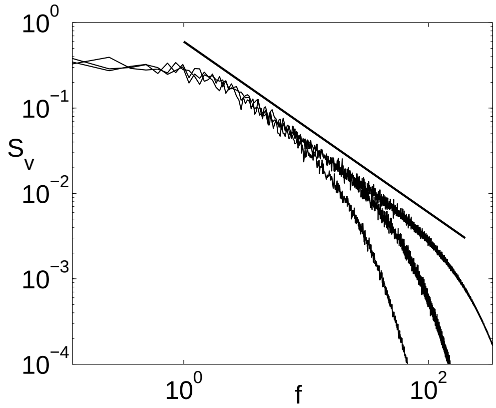

Fig. 3 – Spectrum of the phase speed $\dot{\theta}_N$ of the N-th oscillator (for $N = 50$), with non-uniform couplings as in eq. (11): $K_1 = K_2 = \ldots = K_{N-1} = 0.3$, and for three different values of the strong coupling: $K_N = 10^3$ (bottom spectrum), $K_N = 10^4$, and $K_N = 10^5$ (top spectrum). Averages are taken over 100 realizations.

case as $X(t) = \bar{X} + x(t)$, $Y(t) = \bar{Y} + y(t)$, where the bar denotes the $N \to \infty$ limit, and $x(t)$, $y(t)$ are the finite-size fluctuations. Daido's numerical results for $N > 1000$ indicate that $x(t)$ and $y(t)$ are independent Gaussian processes with variances scaling as N^{-1} with increasing N: see, for example, fig. 7(a) and the associated discussion in [13]. For coupling strengths below the synchronization threshold, $K < K_c$, Kuramoto's analysis yields $\bar{X} = \bar{Y} = 0$, and so the mean-field phase speed is $v(t) = \mathrm{d}/\mathrm{d}t\left[\tan^{-1}(y/x)\right]$. The argument leading to eq. (7) then applies, with $R(\tau)$ being the autocorrelation function of $x(t)$ or $y(t)$. Noting that Daido shows that (for large N) $R(\tau)$ becomes independent of N, we conclude that $v(t)$ has a $1/f$ spectrum as predicted by (9) for all coupling strengths up to the critical value K_c, and that the spectrum is independent of N.

When the coupling strength is above threshold, $K > K_c$, the $N \to \infty$ limit for the mean field has non-zero order parameter ρ. This implies that the mean-field trajectory moves on a limit cycle of radius ρ, while subject to fluctuations whose variance scales as N^{-1}. As noted above, the $1/f$ spectrum in the phase speed may be explained by the large intermittent spikes in its time series. These occur when the mean field $X(t) + iY(t)$ moves close to the origin (see eq. (3)) as happens frequently when $\rho = 0$. However, for non-zero order parameter the relative size of the fluctuations decreases with increasing N, and the mean field is increasingly less likely to move off the limit cycle to get close to the origin. We conclude that for $K > K_c$ the mean-field phase speed does not have a $1/f$ spectrum when N is sufficiently large. However, our numerical results (fig. 1(c), fig. 2) show that an appreciable $1/f$ scaling regime still exists for the super-threshold value $K = 1.9$ when $N = 50$, and indeed it is only for K values in excess of 2 that all traces of $1/f$ scaling disappear in fig. 2. This indicates that the $N \to \infty$ asymptotic behaviour requires $N \gg 50$ for $K = 1.9$, and in particular it signifies that $1/f$ spectra of the mean field phase speed should be measurable in experiments with finitely many oscillators, such as the $N = 64$ example of Kiss *et al.* [8].

Finally, we note that the uncoupled limit suggests the following simple algorithm for simulating $1/f$ noise with low-frequency cutoff ω_0. Let ω_j, $j = 1, \ldots, N$, be random numbers drawn from a Gaussian distribution of mean zero and variance ω_0^2. Then the real part of $\sum_{j=1}^{N} \omega_j e^{iw_j t} / \sum_{j=1}^{N} e^{iw_j t}$ gives the phase speed of the $K = 0$ mean field, and so generates a $1/f$ spectrum as shown by the solid curve in fig. 2.

In summary, we have shown that the mean-field phase speed of coupled oscillators exhibits a $1/f$ spectrum for coupling strengths up to, and somewhat above, the synchronization threshold. The analytical result (10) for the $1/f$ spectrum in the uncoupled limit is supplemented by numerical evidence for $K > 0$. The $1/f$ spectra are unaffected by increasing N when K is sub-threshold; above threshold, however, the spectra are seen only for K sufficiently close to K_c. Our numerical results for the experimentally-feasible value $N = 50$ indicate that $1/f$ phase noise can be seen when the order parameter ρ is as large as 0.7, corresponding to limit cycle behavior such as that in figure 1(c).

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