§4 The Fokker-Planck Equation

• The Fokker-Planck (FPE) for the PDF $P(x, t)$ (or the transition probability $P(x, t|x_0, 0)$) of the process $x(t)$ determined by the Langevin equation (SDE)

$$\frac{dx}{dt} = f(x) + g(x)\eta(t)$$

for a Gaussian zero-mean white noise process $\langle \eta(t) \rangle = 0$, $\langle \eta(t_1)\eta(t_2) \rangle = 2\kappa\delta(t_1 - t_2)$ is

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} \left[ f(x)P \right] + \kappa \frac{\partial}{\partial x} \left[ g(x) \frac{\partial}{\partial x} (g(x)P) \right].$$

• Note: this uses the Stratonovich interpretation of white noise, see later notes on the Itô/Stratonovich dilemma.

• **Proof of FPE**

• Two steps: first, derive Kramers Moyal expansion [Risken 4.1], then calculate the drift and diffusion coefficients.

• The PDF satisfies

$$P(x, t + \tau) = \int P(x, t + \tau|x', t)P(x', t)dx'$$

• Define the moments (using notation $\xi(t)$ for the process path $x(t)$ to avoid confusion with integration variables):

$$M_n(y, t, \tau) = \langle [\xi(t + \tau) - \xi(t)]^n \rangle_{\xi(t)=y} = \int (x-y)^n P(x, t + \tau|y, t) \, dx = \int \Delta^n P(y + \Delta, t + \tau|y, t) \, d\Delta,$$

where $\Delta = x - y$.

• The integrand of (2) can be expanded in a Taylor series for small $\Delta = x - x'$ as

$$P(x, t + \tau|x', t)P(x', t) = P(x + \Delta - \Delta, t + \tau|x - \Delta, t)P(x - \Delta, t)$$

$$= \sum_{n=0}^{\infty} \frac{(-\Delta)^n}{n!} \left( \frac{\partial}{\partial x} \right)^n [P(x + \Delta, t + \tau|x, t)P(x, t)].$$
• Put this under the integral in (2) to get (exchanging summation and integration):
\[ P(x, t + \tau) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{\partial}{\partial x} \right)^n \left[ \left( \int \Delta^n P(x + \Delta, t + \tau | x, t) d\Delta \right) P(x, t) \right] \]
\[ \quad = \sum_{n=0}^{\infty} \left( -\frac{\partial}{\partial x} \right)^n \left[ \frac{M_n(x, t, \tau)}{n!} P(x, t) \right] \quad \text{from (3).} \]

• Note that for the \( n = 0 \) term of the sum we have \( M_0 = 1 \), and so
\[ P(x, t + \tau) = P(x, t) + \sum_{n=1}^{\infty} \left( -\frac{\partial}{\partial x} \right)^n \left[ \frac{M_n(x, t, \tau)}{n!} P(x, t) \right]. \quad (4) \]

• Now assume the moments \( M_n(x, t, \tau) \) can be expanded into Taylor series with respect to \( \tau \), so
\[ \frac{M_n(x, t, \tau)}{n!} = D^{(n)} \tau + O(\tau^2) \quad (5) \]
for small \( \tau \).

• The term with \( \tau^0 \) must vanish, as for \( \tau = 0 \) the transition probability \( P(x, t + \tau | x', t) \) is \( \delta(x - x') \), which has zero moments (3).

• Thus we have, using (5) in (4) (up to linear terms in \( \tau \)) and expanding
\[ P(x, t + \tau) \approx P(x, t) + \tau \frac{\partial P}{\partial t} + O(\tau^2) \]
the result
\[ \frac{\partial P(x, t)}{\partial t} = \sum_{n=1}^{\infty} \left( -\frac{\partial}{\partial x} \right)^n \left[ D^{(n)}(x, t) P(x, t) \right] \]
\[ \quad = -\frac{\partial}{\partial x} \left[ D^{(1)}(x, t) P \right] + \frac{\partial^2}{\partial x^2} \left[ D^{(2)}(x, t) P \right] - \frac{\partial^3}{\partial x^3} \left[ D^{(3)}(x, t) P \right] + \ldots. \quad (6) \]
This is called the **Kramers Moyal expansion**.

• If \( x(t) \) is a Markov process (as it is if it solves the white-noise Langevin equation (1) for instance), then the value of the coefficients \( D^{(n)} \) at time \( t \) do not depend on the the earlier values of \( x(t') \) for \( t' < t \) except through \( x(t) \). In this case equation (6) is first-order in time and its solution is determined uniquely by the initial distribution \( P(x, 0) \) and the appropriate boundary conditions.

• **Step 2 of proof**: Determining the coefficients \( D^{(1)} \) (“drift coefficient”), \( D^{(2)} \) (“diffusion coefficient”), and \( D^{(n)} \) for \( n \geq 3 \) for (6), using the Langevin equation (1).
• We require the coefficients $D^{(n)}(x, t)$ defined as (see (5):

$$D^{(n)}(x, t) = \lim_{\tau \to 0} \frac{1}{\tau^n} \frac{M_n(x, t, \tau)}{n!}$$

$$= \frac{1}{n!} \lim_{\tau \to 0} \frac{1}{\tau^n} \langle (\xi(t + \tau) - x)^n \rangle|_{\xi(t) = x},$$

using $\xi(t)$ for the path of the process $x(t)$.

• But $\xi(t)$ is $x(t)$, which is the solution of the Langevin equation

$$\frac{dx}{dt} = f(x) + g(x)\eta(t),$$

so we have the integral equation form:

$$\xi(t + \tau) = x + \int_t^{t+\tau} [f(\xi(t')) + g(\xi(t'))\eta(t')] dt'.$$

• Expand $f$ and $g$ about $\xi(t') = x$:

$$f(\xi(t')) = f(x) + f'(x)(\xi(t') - x) + \ldots$$

$$g(\xi(t')) = g(x) + g'(x)(\xi(t') - x) + \ldots$$

• This gives

$$\xi(t + \tau) - x = \tau f(x) + f'(x) \int_t^{t+\tau} [\xi(t') - x] dt' + \ldots$$

$$+ g(x) \int_t^{t+\tau} \eta(t') dt' + g'(x) \int_t^{t+\tau} [\xi(t') - x] \eta(t') dt' + \ldots \quad (7)$$

• To deal with the $\xi(t') - x$ in the integrands, we iterate (7), producing

$$\tau f(x) + f'(x) \int_t^{t+\tau} dt' \left[ (t' - t)f(x) + g(x) \int_t^{t'} \eta(t'') dt'' \right] + \ldots$$

$$+ g(x) \int_t^{t+\tau} \eta(t') dt' + g'(x) \int_t^{t+\tau} dt' \left[ (t' - t)f(x) + g(x) \int_t^{t'} \eta(t'') dt'' \right] \eta(t') + \ldots \quad (8)$$

• By repeated iteration we generate an infinite series involving integrals of $f$, $g$, and their derivatives at $x$, plus the white noise $\eta$ at various times. If we average (8) we
\[ \langle \xi(t + \tau) - x \rangle = \tau f(x) + f'(x)f(x)\frac{1}{2}\tau^2 + \ldots \]
\[ + g(x)0 + g'(x)\int_{t}^{t+\tau} dt' \left[ 0 + g(x)\int_{t}^{t'} dt'' \langle \eta(t'')\eta(t') \rangle \right] + \ldots \]
\[ = \tau f(x) + \frac{1}{2}\tau^2 f'(x)f(x) + \ldots \]
\[ + g'(x)g(x)2\kappa \int_{t}^{t+\tau} dt' \int_{t}^{t'} dt'' \delta(t'' - t') + \ldots \]

- Note that the double integral may be evaluated by setting \( s' = t' - t \), \( s'' = t'' - t \) to obtain
\[ \int_{0}^{\tau} ds' \int_{0}^{s'} ds'' \delta(s'' - s') = \frac{1}{2} \int_{0}^{\tau} ds' \int_{0}^{s'} ds'' \delta(s'' - s') = \frac{1}{2}\tau. \]

- Dividing \( \langle \xi(t + \tau) - x \rangle \) by \( \tau \) and taking the limit \( \tau \to 0 \) we get
\[ D^{(1)}(x, t) = \lim_{\tau \to 0} \frac{1}{\tau} \langle (\xi(t + \tau) - x) \rangle \]
\[ = f(x) + \kappa g(x)g'(x). \]

All higher order terms in \( \tau \)-expansion go to zero, see [Risken p. 50].

- Similarly, the diffusion coefficient is found to be
\[ D^{(2)}(x, t) = \frac{1}{2} \lim_{\tau \to 0} \frac{1}{\tau} g(x)^2 \int_{t}^{t+\tau} dt' \int_{t}^{t+\tau} dt'' 2\kappa \delta(t'' - t') \]
\[ = \kappa g(x)^2, \]
and all higher coefficients \( D^{(n)} \) for \( n \geq 3 \) are zero.

- **Putting steps together:** The Kramers-Moyal expansion (6) for the case where \( x(t) \) is the solution of (1) gives the Fokker-Planck equation for the PDF \( P(x, t) \):
\[ \frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} \left[ (f(x) + \kappa g(x)g'(x)) P \right] + \kappa \frac{\partial^2}{\partial x^2} \left[ g(x)^2 P \right]. \]
Noting the identities

\[-\kappa gg' P + \kappa \frac{\partial}{\partial x} \left[ g^2 P \right] = \kappa gg' P + \kappa g^2 \frac{\partial P}{\partial x} = \kappa g \frac{\partial}{\partial x} (gP),\]

we have the equivalent form of the FPE

\[\frac{\partial P}{\partial t} = - \frac{\partial}{\partial x} \left[ f(x) P \right] + \kappa \frac{\partial}{\partial x} \left[ g(x) \frac{\partial}{\partial x} (g(x) P) \right],\]

as claimed. [End of proof.]

Note: this form of the FPE also applies if \(f\) and \(g\) depend deterministically on time, see Risken for details.

The derivation we have used is referred to as the “Stratonovich interpretation” of white noise. As we will discuss later, an alternative interpretation due to Itō leads to the so-called Itō form of the Fokker-Planck equation:

\[\frac{\partial P}{\partial t} = - \frac{\partial}{\partial x} \left[ f(x) P \right] + \kappa \frac{\partial^2}{\partial x^2} \left[ g(x)^2 P \right].\]

The Stratonovich and Itō interpretations coincide if \(g(x) \equiv \text{constant} : \) this case is called “additive” (as opposed to “multiplicative”) noise. We will look at some examples of the FPE, keeping \(g\) constant until we address the Itō/Stratonovich dilemma.

Fokker-Planck equation in higher dimensions.

If the Langevin equation

\[\frac{dx}{dt} = f(x) + g(x) \eta(t)\]

is for vector-valued \(x(t)\) (e.g. \(x(t) \in \mathbb{R}^n\)), we write the components \(x = (x_1, x_2, \ldots, x_n)\), and the equations as

\[\frac{dx_i}{dt} = f_i(x) + g_i(x) \eta_i(t) \quad \text{for } i = 1, 2, \ldots, n.\]

A similar derivation to the one-dimensional case (see [Risken 4.7]) leads to the multidimensional FPE for the joint PDF \(P(x, t)\):

\[\frac{\partial P}{\partial t} = - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[ D_i^{(1)}(x, t) P(x, t) \right] + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left[ D_{ij}^{(2)}(x, t) P(x, t) \right].\]
• If the noise is additive, we have \( g_i = \text{constant} \) for all \( i \) and
\[
D_i^{(1)} = f_i.
\]

• If the noise components are independent of each other and of equal intensity, i.e.,
\[
\langle \eta_i(t)\eta_j(t') \rangle = 2\kappa \delta_{ij} \delta(t - t'),
\]
then the FPE is often written as the convection-diffusion equation
\[
\frac{\partial P}{\partial t} = -\nabla \cdot (f P) + \kappa \nabla^2 P
\]
where \( \nabla \) and \( \nabla^2 \) are the familiar divergence and Laplacian operators.

**Example:** Brownian motion of dye molecules in an annular flow.

• The two-dimensional position vector \( x(t) \) gives the position of a particle passively advected by the steady fluid flow
\[
\frac{dx}{dt} = v(x).
\]

• Add effect of random molecular collisions:
\[
\frac{dx}{dt} = v(x) + \eta(t),
\]
with \( \langle \eta_i(t)\eta_j(t') \rangle = 2\kappa \delta_{ij} \delta(t - t') \).

• The FPE is
\[
\frac{\partial P}{\partial t} = -\nabla \cdot (v P) + \kappa \nabla^2 P
\]
\[
= -v \cdot \nabla P + \kappa \nabla^2 P,
\]
with the second line following from the assumption that the fluid flow is incompressible (i.e., \( \nabla \cdot v = 0 \)).

• In cylindrical polar coordinates, with fluid flow only in the angular direction (independent of \( \phi \)), this gives
\[
\frac{\partial P}{\partial t} + \frac{v(r)}{r} \frac{\partial P}{\partial \phi} - \frac{\kappa}{r} \frac{\partial}{\partial r} \left( r \frac{\partial P}{\partial r} \right) - \frac{\kappa}{r^2} \frac{\partial^2 P}{\partial \phi^2} = 0.
\]