Influence of Noise Intensity on the Spectrum of an Oscillator

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***Abstract*—This paper investigates the influence of high-intensity noise on the correlation spectrum of a two-dimensional (2-D) nonlinear oscillator. An exact analytical solution for the correlation spectrum of this 2-D oscillator is provided. The analytical derivations are well suited for oscillators with white noise of any intensity, but computational constraints on the solution of the partial differential equation may make it impractical for cases where the number of state variables exceeds three. The spectral results predicted by our analytical method are verified by numerical simulations of the noisy oscillator in the time domain. We find that the peak of the oscillator spectrum shifts toward higher frequencies as the noise intensity is increased, as opposed to the fixed oscillation frequency predicted in the existing literature. This phenomenon does not appear to have been reported previously in the context of phase noise in oscillators.**

***Index Terms*—Fokker–Planck equation, nonlinear perturbation technique, oscillator, phase noise.**

I. INTRODUCTION

PAPERS by Kaertner [1], [2] and a recent paper by Demir *et al.* [3] use nonlinear perturbation techniques to determine an exact equation for the phase error of any oscillator. Coram [4] presented a simple analytically solvable example of a two-dimensional (2-D) oscillator in order to explain the claims made in [2] and [3] that the noise perturbation in an oscillator must be decomposed into its components along the Floquet vectors of the system and that these Floquet vectors need not be orthogonal. However, the analysis in [2]–[4] assumes low-intensity noise and treats the phase noise problem as a diffusion problem. Our aim is to find an exact analytical spectrum of this 2-D oscillator in the presence of excess white noise. Since perturbation techniques are not applicable in the high-noise regime, we resort back to the Fokker–Planck equation (FPE) corresponding to the stochastic differential equations (SDEs) [5], [6] describing the oscillator by Coram [4], due to the following reasons.

1) The FPE does not depend on the assumption of low-intensity noise, whereas perturbation techniques (linear/nonlinear) need to make this assumption.
2) The phase noise problem is often treated as a diffusion problem [2], [3], [7] as opposed to a more general convection-diffusion problem.

Thus, we consider the FPE resulting from the noisy oscillator equations instead of defining an SDE describing the phase error based on the assumption of pure diffusion. Hence, our approach allows us to determine the exact correlation spectrum for this specific 2-D oscillator case in a more general way than that in the existing literature [2], [3]. We also verify our exact result by solving the nonlinear SDEs in the time domain by using various numerical techniques. In the course of investigating the power spectrum of this 2-D oscillator, we detect a shift in the spectral peak toward higher frequencies, away from the noise-free value, and note that the size of this shift depends on the input noise intensity. This phenomenon of shift in the spectral peak is small at low-noise intensities (which is the case for any practical oscillator described by perturbation technique); however, the shift becomes significant at very high noise intensities, as is shown in this paper.

A. Main Results

Our main results are as follows.

1) We provide an analytical method for determining the exact spectrum of a simple 2-D nonlinear oscillator in the presence of excess white noise.
2) We find that the peak of the oscillator spectrum shifts toward higher frequencies as the intensity of the noise is increased and that the size of the shift depends on the noise intensity.
3) We demonstrate by example that the phase noise problem in oscillators is a convection-diffusion problem.

II. PHASE NOISE SPECTRUM OF A 2-D OSCILLATOR BY APPLYING NONLINEAR PERTURBATION TECHNIQUE

Consider the simple case of a nonlinear 2-D oscillator by Coram [4] in (r, θ) coordinates

$$\frac{dr}{dt} = r - r^2 + \beta b_r(t) \tag{1}$$

$$\frac{d\theta}{dt} = 1 + r + \beta b_\theta(t) \tag{2}$$

where β is a constant denoting the intensity of noise and $b_r(t)$ and $b_\theta(t)$ are white Gaussian noise processes. Transforming (1) and (2) to Cartesian coordinates, we get

$$\dot{x} = x - y - (x+y)\sqrt{x^2+y^2} + \beta b_1(t) \tag{3}$$

$$\dot{y} = x + y + (x-y)\sqrt{x^2+y^2} + \beta b_2(t) \tag{4}$$

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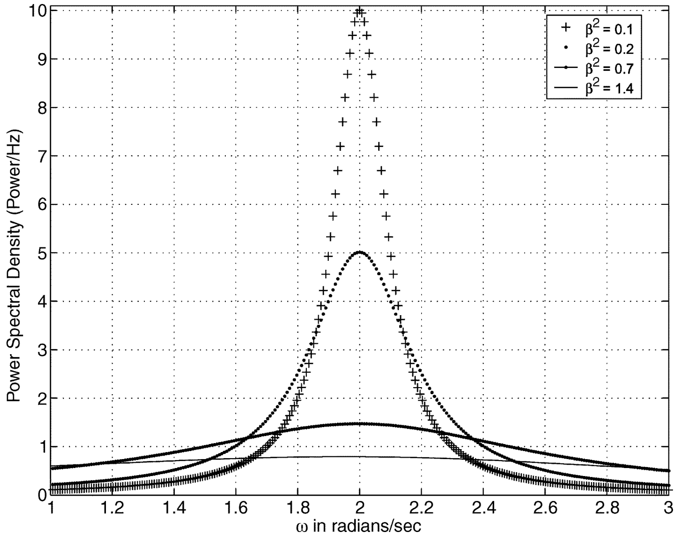

Fig. 1. Phase noise spectrum of Coram's oscillator by applying Demir's method for different values of noise power, $\beta^2 = [0.1, 0.2, 0.7, 1.4]$ or SNR $SNR = [6.99\text{ dB}, 3.98\text{ dB}, -1.46\text{ dB}, -4.47\text{ dB}]$. Note that the position of the spectral peak does not depend on β.

where $b_1(t)$ and $b_2(t)$ are Gaussian noise processes in the new coordinates. In compact form, we can rewrite (3) and (4) as

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x}) + \mathbf{B}(\mathbf{x}, t)\mathbf{b}(t). \quad (5)$$

As described by Kaertner [2] and Demir *et al.* [3], we can decompose the process $\mathbf{x}(t)$ of (5) into the limit cycle component $x^0(t)$ and a component perpendicular to the limit cycle $\Delta x_\perp(t)$. Thus, we can write the correlation matrix as follows:

$$R_{xx}(t,\tau) = R_{x^0x^0}(\tau) + R_{x^0\Delta x_\perp}(\tau) + R_{\Delta x_\perp x^0}(\tau) + R_{\Delta x_\perp \Delta x_\perp}(\tau). \quad (6)$$

The above equation describes the phase noise, the correlations between phase and amplitude noise, and the amplitude noise of the stationary stochastic process $\mathbf{x}(t)$. The phase-noise power spectrum is the Fourier transform of the first term of (6). Thus, the spectrum resulting from [3, eq. (35)] when applied to the oscillator described by (1) and (2) is given by

$$S(\omega) = \frac{1}{2}\frac{\omega_0^2 c}{\frac{1}{4}\omega_0^4 c^2 + (\omega - \omega_0)^2} + \frac{1}{2}\frac{\omega_0^2 c}{\frac{1}{4}\omega_0^4 c^2 + (\omega + \omega_0)^2} \quad (7)$$

where $\omega_0 = 2$ is the steady-state oscillator frequency and $\mathrm{c} = 0.5\beta^2$, as given by [3, eq. (44)].

Fig. 1 shows the phase noise spectra for various noise powers β^2 or signal-to-noise ratios (SNRs) ($SNR = 10\log_{10}\{1/2\beta^2\}$) by applying the methods of Kaertner [2] and Demir *et al.* [3] to the 2-D oscillator. Note that the position of the spectral peak does not depend on β.

III. Exact Spectrum for the Simple 2-D Oscillator

The FPE [6], [8] corresponding to (5) can be written in polar coordinates as

$$\frac{\partial P}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}(rFP) - \frac{\partial}{\partial \theta}(MP) + \frac{\beta^2}{2r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right) + \frac{\beta^2}{2r^2}\frac{\partial^2 P}{\partial \theta^2} \quad (8)$$

where $F = r - r^2$, $M = 1 + r$, and $P = P(r, \theta, t|r_0, \theta_0, t_0)$ is the transition probability.

We next calculate the stationary distribution $P_\infty(r)$. The fact that F and M are independent of the angle θ simplifies the analysis. Setting derivatives with respect to t and θ to zero in (8) gives us an ordinary differential equation for P_∞

$$\frac{1}{r}\frac{d}{dr}\left(r(r - r^2)P_\infty\right) - \frac{\beta^2}{2r}\frac{d}{dr}\left(r\frac{dP_\infty}{dr}\right) = 0. \quad (9)$$

Equation (9) has a normalizable solution

$$P_\infty(r) = C\exp\left[\frac{2}{\beta^2}\left(\frac{r^2}{2} - \frac{r^3}{3}\right)\right] \quad (10)$$

with constant C given by the requirement that the 2-D phase-space integral over P_∞ gives unity, i.e.,

$$\int_0^\infty P_\infty(r)2\pi r dr = 1. \quad (11)$$

This implies that C is given by

$$C = \frac{1}{\left(2\pi\int_0^\infty \mathrm{r}\exp\left[\frac{2}{\beta^2}\left(\frac{r^2}{2} - \frac{r^3}{3}\right)\right]dr\right)}. \quad (12)$$

The normalized $P_\infty(r)$ is plotted against the time-domain histogram of the steady-state radius for two values of noise power (β^2) and shown in Figs. 2 and 3.

The steady-state histogram of the radius is obtained by numerically integrating (3) and (4) simultaneously (using the initial conditions of $x_0 = 1$ and $y_0 = 0$) in Matlab [12] for a period of $T = 256\pi$ seconds involving 300 realizations of 2^{16} points each and then concatenating the last 500 points from each of those realizations to generate 150 000 points.

In order to determine the correlation function $R(\tau)$, we apply **Theorem 1** (given in the Appendix) in polar coordinates, which gives us the following partial differential equation by Gleeson *et al.* [9]:

$$\frac{\partial Q_i}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left[r(r - r^2)Q_i\right] - \frac{1}{r}\frac{\partial}{\partial \theta}\left[(1 + r)Q_i\right] + \frac{\beta^2}{2r}\frac{\partial}{\partial r}\left[r\frac{\partial Q_i}{\partial r}\right] + \frac{\beta^2}{2r^2}\frac{\partial^2 Q_i}{\partial \theta^2} \quad (13)$$

where $i = 1, 2$, and the vector $Q = [Q_1, Q_2]^T$ is a solution to the above PDE given the initial conditions. The correlation function is then derived from the correlation tensor $R_{ij}(\tau)$ as

$$R(\tau) = \sum_{i=1}^{2}\sum_{j=1}^{2} R_{ij}(\tau) \quad (14)$$

where $R_{ij}(\tau)$ depends on Q (as shown in the Appendix). Equation (14) is Fourier-transformed to yield the oscillator spectrum [8], [10]

$$S(\omega) = \frac{1}{R(0)}\int_{-\infty}^{\infty} \exp(j\omega\tau)R(\tau)\,d(\tau). \quad (15)$$

In Figs. 4 and 5, we overlay our exact analytical spectra and our estimated correlation spectra derived from finite time-domain simulations for two different values of noise power (β^2).

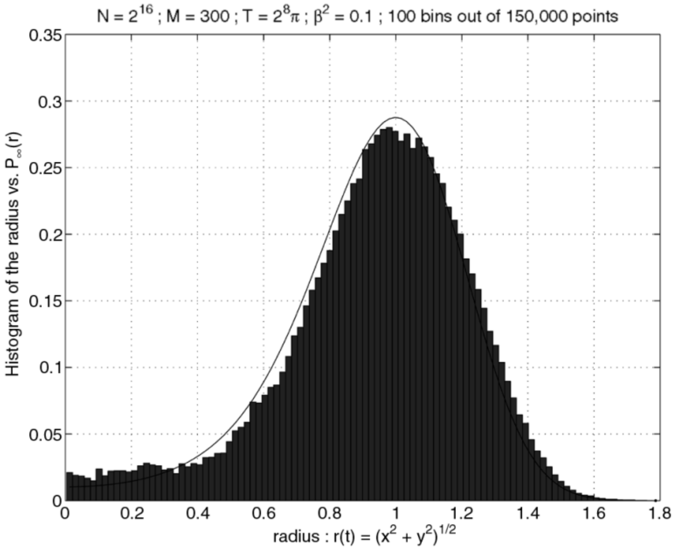

Fig. 2. Exact normalized pdf $P_\infty(r)$, versus histogram of the radius when $\beta^2 = 0.1$ or $SNR = 6.99$ dB.

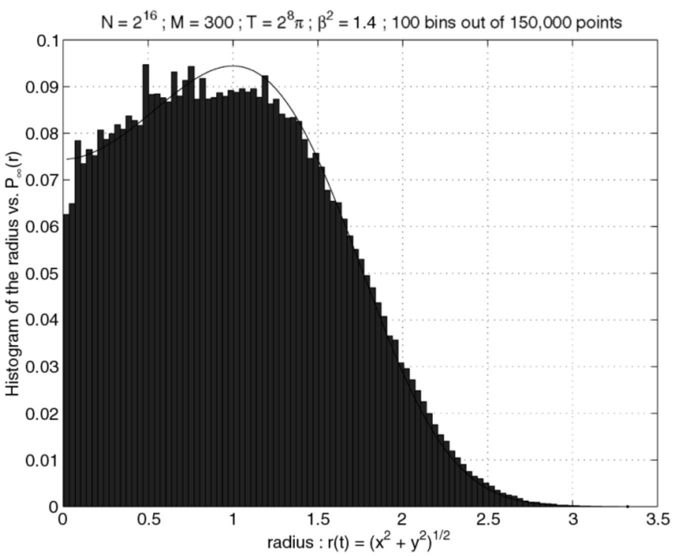

Fig. 3. Exact normalized pdf $P_\infty(r)$, versus histogram of the radius when $\beta^2 = 1.4$ or $SNR = -4.47$ dB.

We estimate the power spectral density (psd) of our data using Welsh's averaged modified periodogram method [10], [11] by integrating the SDEs (3) and (4) starting with the steady-state initial conditions of $(x_0, y_0) = (1, 0)$ for a total integration time of $T = 2^{12}\pi$ s (π being the oscillator's period) to generate $N = 2^{20}$ points.

Figs. 4 and 5 show that, as we increase the noise intensity, the spectral peak of the oscillator tends to shift toward higher frequencies.

These spectra are qualitatively different from those predicted by Kaertner [2] and Demir *et al.* [3] in Fig. 1 for the case of low-noise intensities. In particular, our approach predicts that the spectral peak moves to higher frequencies as the noise level increases and that this shift increases with the intensity of white noise and is independent of the regime of operation i.e., low-noise or high-noise. The reason for such a difference may be traced to the fact that the amplitude contribution of the noise has

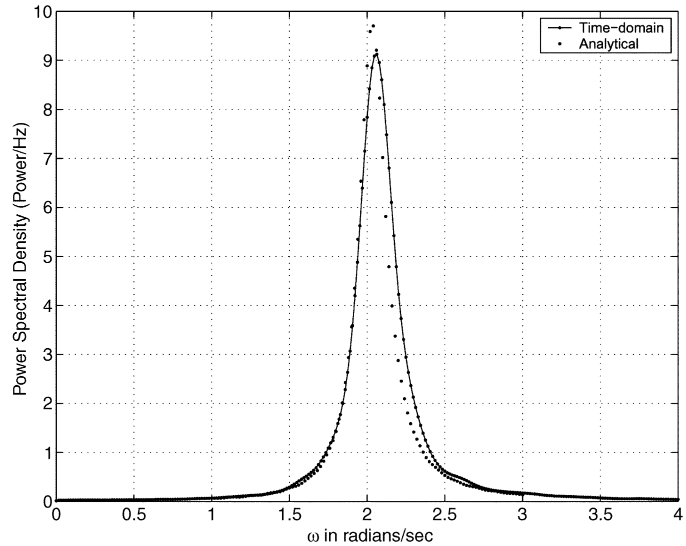

Fig. 4. Exact analytical spectrum (dotted curve) versus estimated spectrum (solid curve) by averaging the modified periodogram method. Noise power: $\beta^2 = 0.1$ or $\mathrm{SNR} = 6.99$ dB. A Welsh window of 2^{13} points is used with an overlap of 2^{12} points.

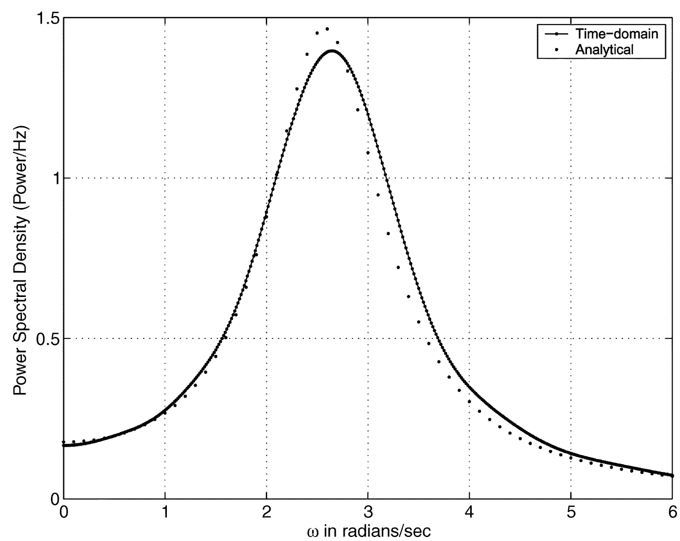

Fig. 5. Exact analytical spectrum (dotted curve) versus estimated spectrum (solid curve) by averaging modified periodogram method. Noise power: $\beta^2 = 1.4$ or $\mathrm{SNR} = -4.47$ dB. A Welsh window of 2^{10} points is used with an overlap of 2^9 points.

been neglected in Kaertner [2] and Demir *et al.* [3], whereas our analytical technique does not neglect the amplitude contribution of the noise and hence gives the correct spectrum for this simple 2-D oscillator.

IV. Conclusion

We have described a method for calculating the exact analytical spectra of a 2-D oscillator in the presence of white noise of arbitrary intensity and have verified our result by successfully simulating in Matlab a set of nonlinear SDEs that describe this noisy 2-D oscillator.

The nonlinear perturbation technique is shown to yield a good approximation to the exact spectrum when the noise intensity is low, as expected, but fails to capture the shift of the spectral peak toward higher frequencies, away from the noise-free value. This shift is shown to depend on the intensity of noise in the system. For higher noise intensities, where the perturbation technique is not applicable, our analytical scheme is able to predict the correct spectrum for this simple 2-D oscillator. We note that, although the technique described here is useful for the simple 2-D example described by Coram, the problem of solving the FP equation for systems of order greater than three is nontrivial.

APPENDIX

Theorem 1: Let the vector $Q(x,\tau)$ be the solution of the partial differential equation (repeated indices imply summation over the n spatial dimensions)

$$\frac{\partial Q_i}{\partial \tau}+\frac{\partial}{\partial x_j}(f_j Q_i)-\frac{\beta^2}{2}\frac{\partial^2 Q_i}{\partial x_j \partial x_j}=0, \quad i=1,\ldots,n \tag{16}$$

with initial conditions

$$Q_i(x,0)=P_\infty(x)x_i, \qquad i=1,2,\ldots,n. \tag{17}$$

Then, the correlation tensor

$$R_{ij}(\tau)=\langle x_i(t)x_j(t+\tau)\rangle \tag{18}$$

may be calculated as

$$R_{ij}(\tau)=\int x_j Q_i(x,\tau)\,dx \tag{19}$$

where the integral is over the n-dimensional phase space. The oscillator spectrum can be determined by first finding the correlation function which is given as

$$R(\tau)=\sum_{i=1}^{n}\sum_{j=1}^{n}R_{ij}(\tau) \tag{20}$$

and then taking the Fourier transform of the correlation function

$$S(\omega)=\frac{1}{R(0)}\int_{\infty}^{\infty}\exp(j\omega\tau)R(\tau)\,d\tau. \tag{21}$$

Proof: The proof is based on the fact that the stochastic process x is a Markov process, and so its correlation function may be represented as

$$\langle x_i(t)x_j(t+\tau)\rangle=\int dy P_\infty(y)y_i\int dx P(x,\tau|y,0)x_j \tag{22}$$

where $P(x,\tau|y,0)$ is the transition probability, i.e., the probability that the process has value x at time τ, given that it had value y at time $\tau=0$. The transition probability is governed by the (scalar) FPE

$$\frac{\partial P}{\partial \tau}+\frac{\partial}{\partial x_j}(f_j P)-\frac{\beta^2}{2}\frac{\partial^2 P}{\partial x_j \partial x_j}=0 \tag{23}$$

with initial condition

$$P(x,0|y,0)=\delta(x-y). \tag{24}$$

Now define the vector Q by

$$Q_i(x,\tau)=\int P_\infty(y)y_i P(x,\tau|y,0)dy, \qquad \text{for } i=1,\ldots,n. \tag{25}$$

The theorem follows from multiplying (23) and (24) by $P_\infty(y)y_i$ and integrating over y. Applying the above theorem for the vector $Q=[Q_1,Q_2]^T$ in polar coordinates, we get the following partial differential equation:

$$\frac{\partial Q_i}{\partial \tau}+\frac{1}{r}\frac{\partial}{\partial r}\left[r(r-r^2)Q_i\right]+\frac{1}{r}\frac{\partial}{\partial \theta}[(1+r)Q_i] -\frac{\beta^2}{2r}\frac{\partial}{\partial r}\left[r\frac{\partial Q_i}{\partial r}\right]-\frac{\beta^2}{2r^2}\frac{\partial^2 Q_i}{\partial \theta^2}=0 \tag{26}$$

for $i=1,2$. The initial conditions, from (17), are

$$\begin{aligned} Q_1(\tau=0)&=P_\infty(r)r\cos(\theta)\\ Q_2(\tau=0)&=P_\infty(r)r\sin(\theta). \end{aligned} \tag{27}$$

For the 2-D oscillator of Coram [4], where the radial velocity F and the azimuthal velocity M are independent of θ, the solutions may be found in the form

$$Q_i(r,\theta,\tau)=G_i(r,\tau)\cos\theta+H_i(r,\tau)\sin\theta, \quad i=1,2. \tag{28}$$

Inserting the $i=1$ term of (28) into (26) leads to a pair of coupled PDEs for G_1 and H_1

$$\begin{aligned} &\frac{\partial G_1}{\partial \tau}+\frac{1}{r}\frac{\partial}{\partial r}\left[r(r-r^2)G_1\right]+(1+r)H_1 -\frac{\beta^2}{2r}\frac{\partial}{\partial r}\left[r\frac{\partial G_1}{\partial r}\right]+\frac{\beta^2}{2r^2}G_1=0\\ &\frac{\partial H_1}{\partial \tau}+\frac{1}{r}\frac{\partial}{\partial r}\left[r(r-r^2)H_1\right]-(1+r)G_1 -\frac{\beta^2}{2r}\frac{\partial}{\partial r}\left[r\frac{\partial H_1}{\partial r}\right]+\frac{\beta^2}{2r^2}H_1=0 \end{aligned} \tag{29}$$

with initial conditions

$$\begin{aligned} G_1(\tau=0)&=P_\infty(r)r\\ H_1(\tau=0)&=0. \end{aligned} \tag{30}$$

The appropriate boundary conditions are

$$\begin{aligned} G_1(r=0)&=H_1(r=0)=0\\ G_1(r\to\infty)&=H_1(r\to\infty)=0. \end{aligned} \tag{31}$$

A similar pair of PDEs may be found by inserting the $i=2$ term of (28) into (26), but with initial conditions

$$\begin{aligned} G_2(\tau=0)&=0\\ H_2(\tau=0)&=P_\infty(r)r. \end{aligned} \tag{32}$$

Noting the symmetry of equations and initial conditions, we can immediately conclude that $G_1=H_2$ and $G_2=-H_1$. Assuming that the G and H functions have been found, the correlation tensor is calculated from (19) and then the correlation function given by (20) to yield

$$\begin{aligned} R(\tau)&=\sum_{i=1}^{2}\sum_{j=1}^{2}R_{ij}(\tau)\\ &=\int_0^\infty\int_0^{2\pi}\left[(G_1+G_2)\cos^2(\theta) +(H_1+H_2)\sin^2(\theta)\right]r^2 d\theta\,dr\\ &=2\pi\int_0^\infty G_1 r^2 dr. \end{aligned} \tag{33}$$

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