First passage time problems

- The first passage time is the time at which the stochastic variable \( x(t) \) (restrict to 1D here) first leaves a specified domain, e.g. by crossing a “barrier” level.
- Suppose the initial condition is each realization is \( x(0) = x' \).
- The first passage time \( T \) in a given realization is the time when \( x(t) \) reaches a boundary for the first time: this is a random variable, and depends on \( x' \).
- If both boundaries \( x_1 \) and \( x_2 \) \((x_2 > x_1)\) are absorbing, then either
  \[
  x(T) = x_1
  \]
  or
  \[
  x(T) = x_2.
  \]
- If one boundary is reflecting, only the other boundary need be considered.
- We want to calculate the distribution function for the random variable \( T \).
- Consider the PDF \( P(x, t|x', 0) \) for \( x(t) \) starting with \( x(0) = x' \).
- When \( x(t) \) reaches a boundary we remove it from the ensemble of realizations. Therefore \( P = 0 \) for \( x \geq x_2 \) and \( x \leq x_1 \). So the boundary condition is that of an absorbing wall.
- The PDF \( P \) satisfies the FPE
  \[
  \frac{\partial P}{\partial t} = L_x P,
  \]
  with initial condition
  \[
  P(x, 0|x', 0) = \delta(x - x') \text{ for } x_1 < x < x_2
  \]
  and boundary conditions
  \[
  P(x, t|x', 0) = 0 \text{ for } x = x_1 \text{ or } x = x_2.
  \]
  The operator \( L_x \) is given by
  \[
  L_x P = -\frac{\partial}{\partial x} (f P) + \frac{\partial^2}{\partial x^2} \left( \frac{g^2}{2} P \right),
  \]
  when \( x(t) \) satisfies the Itô SDE
  \[
  dx = f(x)dt + g(x)dW.
  \]
• The probability $W(x', T)$ of a realization starting at $x'$ and not yet reaching either boundary up to time $T$ is

$$W(x', T) = \int_{x_1}^{x_2} P(x, T|x', 0)dx.$$ 

• The fraction of escaped realizations at time $T$ is then

$$1 - W(x', T)$$

and the rate of change of this gives the distribution function $w(T)$ for the first passage time:

$$w(x', T) = -\frac{\partial W(x', T)}{\partial T} = -\int_{x_1}^{x_2} \frac{\partial P}{\partial t}(x, T|x', 0)dx.$$

• Thus if we find $P(x, t|x', 0)$ (by solving the time-dependent FPE) we can get the distribution function for the first passage time of realizations starting at $x'$ by integrating $\frac{\partial P}{\partial t}$ over the domain.

• The moments of $w(x', T)$ are of special interest.

• The mean first passage time (MFPT) $T_1(x')$ of realizations starting from $x'$ is defined as

$$T_1(x') = \int_0^\infty Tw(x', T)dT.$$ 

• Higher moments

$$T_n(x') = \int_0^\infty T^n w(x', T)dT$$

can also be defined [Risken p.180].

• Next, we derive an ODE for $T_1(x')$. Starting from:

$$T_1(x') = \int_0^\infty Tw(x', T)dT$$

$$= -\int_0^\infty t \left( \int_{x_1}^{x_2} \frac{\partial P}{\partial t}dx \right) dt$$

$$= -\int_{x_1}^{x_2} dx \int_0^\infty t \frac{\partial P}{\partial t}dt$$

$$= \int_{x_1}^{x_2} dx \int_0^\infty P(x, t|x', 0)dt \quad \text{by parts}$$

$$= \int_{x_1}^{x_2} p_1(x, x')dx, \quad (1)$$
where \( p_1(x, x') \) is defined as [Risken equation (8.6)]:

\[
p_1(x, x') = \int_0^\infty P(x, t|x', 0) dt.
\]

- We derive an equation for \( p_1 \) by integrating the FPE for \( P \) over all \( t \):
  \[
  \int_0^\infty \frac{\partial P}{\partial t} dt = -\frac{\partial}{\partial x} (fp_1) + \frac{\partial^2}{\partial x^2} \left( \frac{g^2}{2} p_1 \right) = L_x p_1.
  \]

- The left hand side of this can be integrated (assuming \( P \to 0 \) as \( t \to \infty \), so all trajectories eventually escape) to give
  \[
  -P(x, 0|x', 0) = -\delta(x - x').
  \]

- So the equation for \( p_1 \) is
  \[
  L_x p_1 = -\delta(x - x')
  \]  \hspace{1cm} (2)

or, in full:

\[
-\frac{\partial}{\partial x} [fp_1] + \frac{\partial^2}{\partial x^2} \left[ \frac{g^2}{2} p_1 \right] = -\delta(x - x').
\]

Once we solve (2), equation (1) tells us that the MFPT is

\[
T_1(x') = \int_{x_1}^{x_2} p_1(x, x') dx.
\]

- We can express all this as an ODE for \( T_1 \) by using the adjoint operator \( L_x^\dagger \) of \( L_x \). The adjoint of an operator \( L \) is defined by the property
  \[
  \int_{x_1}^{x_2} u(Lv) dx = \int_{x_1}^{x_2} (L^\dagger u) v dx
  \]  \hspace{1cm} (3)

for any functions \( u(x) \) and \( v(x) \) which vanish at the boundaries.

- First, formally write the solution of (2) using the inverse operator \( L_x^{-1} \):
  \[
  p_1 = -L_x^{-1} \delta(x - x').
  \]

- Putting this into (1), we get
  \[
  T_1(x') = -\int_{x_1}^{x_2} L_x^{-1} \delta(x - x') dx.
  \]
This looks like (3) if \( u(x) \equiv 1 \) and \( v(x) = \delta(x - x') \), and letting \( L = L_x^{-1} \), we get from (3):

\[
T_1(x') = - \int_{x_1}^{x_2} (L_x^{-1}) \delta(x - x') dx = -L_x^{-1}.
\]

The inverse operator notation means that \( T_1(x') \) is given by the solution of the ODE

\[
L^\dagger_x T_1(x') = -1
\]

[c.f. Risken (8.15)].

To actually find the adjoint operator we use (3):

\[
\int_{x_1}^{x_2} u \left( L_x v \right) v dx = \int_{x_1}^{x_2} u \left[ -\frac{\partial}{\partial x} (fv) + \frac{\partial^2}{\partial x^2} \left( \frac{g^2}{2} v \right) \right] dx
\]

\[
= \int_{x_1}^{x_2} \left[ \frac{\partial u}{\partial x} fv - \frac{\partial u}{\partial x} \frac{\partial}{\partial x} \left( \frac{g^2}{2} v \right) \right] dx
\]

\[
= \int_{x_1}^{x_2} \left[ f \frac{\partial u}{\partial x} v + \frac{g^2}{2} \frac{\partial^2 u}{\partial x^2} v \right] dx
\]

\[
= \int_{x_1}^{x_2} (L^\dagger_x u) v dx,
\]

and so

\[
L^\dagger_x = f \frac{\partial}{\partial x} + \frac{g^2}{2} \frac{\partial^2}{\partial x^2}.
\]

So the ODE for \( T_1(x') \) is

\[
f(x') \frac{d}{dx'} T_1(x') + \frac{g^2}{2} (x') \frac{d^2}{dx'^2} T_1(x') = -1.
\]

**Example:** Consider GBM with \( x = \ln S \) in the region \( x \in [0, L] \), i.e.,

\[
dx = \sigma dW.
\]

Find the mean first passage time for exit through either \( x = 0 \) or \( x = L \) from an initial position \( x(0) = x' \).

The direct ODE for \( T_1(x') \) is

\[
L^\dagger_x T_1(x') = -1
\]

\[
\Rightarrow \frac{\sigma^2}{2} \frac{d^2}{dx'^2} T_1 = -1,
\]

with boundary conditions \( T_1 = 0 \) at \( x' = 0 \) and at \( x' = L \).
• Solve for $T_1(x')$:

$$T_1(x') = -\frac{1}{\sigma^2} x'^2 + A x' + B,$$

and boundary conditions give $B =$ and

$$A = \frac{1}{\sigma^2} L.$$

• So

$$T_1(x') = \frac{1}{\sigma^2} x'(L - x').$$