Using a table of derivatives

Introduction

In Block 1 you were introduced to the idea of a derivative and you calculated some derivatives from first principles. Rather than always calculate the derivative of a function from first principles it is common practise to use a table of derivatives. This block provides such a table and shows you how to use it.

Prerequisites

Before starting this Block you should...

• understand the meaning of the term ‘derivative’
• understand what is meant by the notation \( \frac{dy}{dx} \)

Learning Outcomes

After completing this Block you should be able to...

✓ use a table of derivatives

Learning Style

To achieve what is expected of you...

分配 sufficient study time

略过 revise the prerequisite material

attempt every guided exercise and most of the other exercises
1. Table of derivatives

Table 1 lists some of the common functions used in engineering and their corresponding derivatives. Remember, that in each case the function in the right-hand column tells us the rate of change, or the gradient of the graph, of the function on the left at a particular value of \(x\).

Table 1
Common functions and their derivatives
(In this table \(k\), \(n\) and \(c\) are constants)

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>constant</td>
<td>0</td>
</tr>
<tr>
<td>(x)</td>
<td>1</td>
</tr>
<tr>
<td>(kx)</td>
<td>(k)</td>
</tr>
<tr>
<td>(x^n)</td>
<td>(nx^{n-1})</td>
</tr>
<tr>
<td>(kx^n)</td>
<td>(knx^{n-1})</td>
</tr>
<tr>
<td>(e^x)</td>
<td>(e^x)</td>
</tr>
<tr>
<td>(e^{kx})</td>
<td>(ke^{kx})</td>
</tr>
<tr>
<td>(\ln x)</td>
<td>(1/x)</td>
</tr>
<tr>
<td>(\ln(kx))</td>
<td>(1/x)</td>
</tr>
<tr>
<td>(\sin x)</td>
<td>(\cos x)</td>
</tr>
<tr>
<td>(\sin(kx + c))</td>
<td>(k \cos(kx + c))</td>
</tr>
<tr>
<td>(\cos x)</td>
<td>(\sin x)</td>
</tr>
<tr>
<td>(\cos(kx))</td>
<td>(\sin(kx))</td>
</tr>
<tr>
<td>(\cos(kx + c))</td>
<td>(\sin(kx + c))</td>
</tr>
<tr>
<td>(\tan x)</td>
<td>(\sec^2 x)</td>
</tr>
<tr>
<td>(\tan(kx))</td>
<td>(k \sec^2(x))</td>
</tr>
<tr>
<td>(\tan(kx + c))</td>
<td>(k \sec^2(kx + c))</td>
</tr>
</tbody>
</table>

Key Point
Particularly important is the rule for differentiating powers of functions.

If \(y = x^n\) then \(\frac{dy}{dx} = nx^{n-1}\)

For example, if \(y = x^3\) then \(\frac{dy}{dx} = 3x^2\).

Example Use Table 1 to find \(\frac{dy}{dx}\) when \(y\) is given by (a) 7\(x\) (b) 14 (c) 5\(x^2\) (d) 4\(x^7\)
Solution
(a) We note that $7x$ is of the form $kx$ where $k = 7$. Using Table 1 we then have $\frac{dy}{dx} = 7$.
(b) Noting that 14 is a constant we see that $\frac{dy}{dx} = 0$.
(c) We see that $5x^2$ is of the form $kx^n$, with $k = 5$ and $n = 2$. The derivative, $knx^{n-1}$, is then $10x$, or more simply, $10x$. So if $y = 5x^2$, then $\frac{dy}{dx} = 10x$.
(d) We see that $4x^7$ is of the form $kx^n$, with $k = 4$ and $n = 7$. Hence the derivative, $\frac{dy}{dx}$, is given by $28x^6$.

Try each part of this exercise
Find $\frac{dy}{dx}$ when $y$ is: (a) $\sqrt{x}$ (b) $\frac{5}{x^3}$ (c) $\frac{7}{x}$

Part (a) Write $\sqrt{x}$ as $x^{\frac{1}{2}}$, and use the result for $x^n$ with $n = \frac{1}{2}$.

Part (b) Write $\frac{5}{x^3}$ as $5x^{-3}$ and use the result for differentiating $kx^n$.

Part (c) Write $\frac{7}{x}$ as $7x^{-1}$.

Try each part of this exercise
Use Table 1 to find $\frac{dz}{dt}$ given: (a) $z = e^t$ (b) $z = e^{8t}$ (c) $z = e^{-3t}$
Although Table 1 is written using $x$ as the independent variable, the Table can be used for any variable.
Part (a) From Table 1, if $y = e^x$, then $\frac{dy}{dx} = e^x$. Hence if $z = e^t$ then $\frac{dz}{dt} = e^t$.

Part (b) Use the result for $e^{kx}$ in Table 1

Part (c) Use the result for $e^{kx}$ in Table 1.

Try each part of this exercise
Find the derivative, $\frac{dy}{dx}$, when $y$ is:- (a) $\sin 2x$ (b) $\cos \frac{x}{2}$ (c) $\tan 5x$

Part (a) Use the result for $\sin kx$ in Table 1.

Part (b) Note that $\cos \frac{x}{2}$ is the same as $\cos \frac{1}{2}x$. Use the result for $\cos kx$ in Table 1, and take $k = \frac{1}{2}$.

Part (c) Use the result for $\tan kx$ in Table 1, and take $k = 5$. 
More exercises for you to try

1. Find the derivative of the following functions:
   (a) $9x^2$  (b) $5$  (c) $6x^3$  (d) $-13x^4$  (e) $\ln 5t$

2. Find $\frac{dz}{dt}$ when $z$ is given by
   (a) $\frac{5}{t^2}$  (b) $\sqrt{t^3}$  (c) $5t^{-2}$  (d) $-\frac{3}{2}t^{\frac{3}{2}}$

3. Find the derivative of each of the following functions
   (a) $\sin 5x$  (b) $\cos 4t$  (c) $\tan 3r$  (d) $e^{2v}$  (e) $\frac{1}{e^{3x}}$

4. Find the derivative of the following
   (a) $\cos \frac{2x}{\pi}$  (b) $\sin(-2x)$  (c) $\tan \pi x$  (d) $e^{\frac{x}{2}}$  (e) $\ln \frac{2}{3}x$

Answer

2. Extending the Table of derivatives

We now quote two simple rules which enable us to extend the range of functions which we can differentiate. The first rule is for differentiating sums, or differences, of functions. The reader should note that all of the rules quoted below can be obtained from first principles using the approach outlined in Block 1.

**Key Point**

The derivative of $f(x) + g(x)$ is $\frac{df}{dx} + \frac{dg}{dx}$

The derivative of $f(x) - g(x)$ is $\frac{df}{dx} - \frac{dg}{dx}$

This rule says that to find the derivative of the sum (or difference) of two functions, we simply calculate the sum (or difference) of the derivatives of each function.

**Example** Find the derivative of $y = x^6 + x^4$.

**Solution**

We simply calculate the sum of the derivatives of each separate function:

$$\frac{dy}{dx} = 6x^5 + 4x^3$$

The second rule tells us how to differentiate a multiple of a function.
Key Point

If \( k \) is a constant the derivative of \( kf(x) \) is \( k \frac{df}{dx} \)

This rule tells us that if a function is multiplied by a constant, \( k \), then the derivative is also multiplied by the same constant, \( k \).

Example Find the derivative of \( y = 8e^{2x} \).

Solution

Here we are interested in differentiating a multiple of the function \( e^{2x} \). We simply differentiate \( e^{2x} \) and multiply the result by 8. Thus

\[
\frac{dy}{dx} = 8 \times 2e^{2x} = 16e^{2x}
\]

Example Find the derivative of each of the following functions.

(a) \( y = 6 \sin 2x \)  (b) \( y = 6 \sin 2x + 3x^2 \)  (c) \( y = 6 \sin 2x + 3x^2 - 5e^{3x} \)

Solution

(a) From Table 1, the derivative of \( \sin 2x \) is \( 2 \cos 2x \). Hence the derivative of \( 6 \sin 2x \) is \( 6(2 \cos 2x) \), that is, \( 12 \cos 2x \).

\[
y = 6 \sin 2x, \quad \frac{dy}{dx} = 6(2 \cos 2x) = 12 \cos 2x
\]

(b) The function is the sum of two terms: \( 6 \sin 2x \) and \( 3x^2 \). Part (a) has already been differentiated to \( 6 \sin 2x \), so we consider the derivative of \( 3x^2 \). The derivative of \( x^2 \) is \( 2x \) and so the derivative of \( 3x^2 \) is \( 3(2x) \), that is, \( 6x \). These derivatives are now summed:

\[
y = 6 \sin 2x + 3x^2 \quad \frac{dy}{dx} = 12 \cos 2x + 6x
\]

(c) We differentiate each part of the function in turn.

\[
y = 6 \sin 2x + 3x^2 - 5e^{3x}
\]

\[
\frac{dy}{dx} = 6(2 \cos 2x) + 3(2x) - 5(3e^{3x})
\]

\[
= 12 \cos 2x + 6x - 15e^{3x}
\]
Try each part of this exercise
Find \( \frac{dy}{dx} \) where \( y = 7x^5 - 3e^{5x} \).

Part (a) The derivative of \( x^5 \) is \( 5x^4 \). Hence obtain the derivative of \( 7x^5 \).

Part (b) Obtain the derivative of \( e^{5x} \).

Part (c) Hence obtain the derivative of \( 3e^{5x} \).

Part (d) So obtain the derivative of \( y = 7x^5 - 3e^{5x} \).

Try each part of this exercise
Find \( \frac{dy}{dx} \) where \( y = 4 \cos \frac{x}{2} + 17 - 9x^3 \).

Part (a) Obtain the derivative of \( \cos \frac{x}{2} \).

Part (b) The derivative of 17 is 0. Now obtain the derivative of \( 9x^3 \).

Part (c) So now obtain the derivative of \( y = 4 \cos \frac{x}{2} + 17 - 9x^3 \).

More exercises for you to try

1. Find \( \frac{dy}{dx} \) when \( y \) is given by:
   (a) \( 3x^7 + 8x^3 \)  
   (b) \( -3x^4 + 2x^{1.5} \)  
   (c) \( \frac{9}{x^7} + \frac{14}{x} - 3x \)  
   (d) \( \frac{3 + 2x}{4} \)  
   (e) \( (2 + 3x)^2 \)

2. Find the derivative of each of the following functions:
   (a) \( z(t) = 5 \sin t + \sin 5t \)  
   (b) \( h(v) = 3 \cos 2v - 6 \sin \frac{v}{2} \)  
   (c) \( m(n) = 4e^{2n} + \frac{2}{e^{2n}} + \frac{n^2}{2} \)  
   (d) \( H(t) = \frac{e^{3t}}{2} + 2 \tan 2t \)  
   (e) \( S(r) = (r^2 + 1)^2 - 4e^{-2r} \)

3. Differentiate the following functions.
   (a) \( A(t) = (3 + e^t)^2 \)  
   (b) \( B(s) = \pi e^{2s} + \frac{1}{2} + 2 \sin \pi s \)  
   (c) \( V(r) = (1 + \frac{1}{2})^2 + (r + 1)^2 \)  
   (d) \( M(\theta) = 6 \sin 2\theta - 2 \cos \frac{\theta}{4} + 2\theta^2 \)  
   (e) \( H(t) = 4 \tan 3t + 3 \sin 2t - 2 \cos 4t \)
3. Evaluating a derivative

The need to find the rate of change of a function at a particular point occurs often. We do this by finding the derivative of the function, and then evaluating the derivative at that point. When taking derivatives of trigonometric functions, any angles must be measured in radians.

Consider a function, \( y(x) \). We use the notation \( \frac{dy}{dx}(a) \) or \( y'(a) \) to denote the derivative of \( y \) evaluated at \( x = a \). So \( y'(0.5) \) means the value of the derivative of \( y \) when \( x = 0.5 \).

**Example** Find the value of the derivative of \( y = x^3 \) where \( x = 2 \). Interpret your result.

**Solution**

We have \( y = x^3 \) and so \( \frac{dy}{dx} = 3x^2 \).

When \( x = 2 \), \( \frac{dy}{dx} = 3(2)^2 = 12 \), that is, \( \frac{dy}{dx}(2) = 12 \) (Equivalently, \( y'(2) = 12 \)).

The derivative is positive when \( x = 2 \) and so \( y \) is increasing at this point. When \( x = 2 \), \( y \) is increasing at a rate of 12 vertical units per horizontal unit.

**More exercises for you to try**

1. Calculate the derivative of \( y = x^2 + \sin x \) when \( x = 0.2 \).
2. Calculate the rate of change of \( i(t) = 4 \sin 2t + 3t \) when
   (a) \( t = \frac{\pi}{4} \) (b) \( t = 0.6 \)
3. Calculate the rate of change of \( F(t) = 5 \sin t - 3 \cos 2t \) when
   (a) \( t = 0 \) (b) \( t = 1.3 \)

[Answer]
End of Block 11.2
\[
\frac{dy}{dx} = nx^{n-1} = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}}. \text{ This may be written as } \frac{1}{2\sqrt{x}}.
\]
\[ 5(-3)x^{-3-1} = -15x^{-4} \]
\[ 7(-1)x^{-1-1} = -7x^{-2} = -\frac{7}{x^2} \]
$8e^{st}$
\[-3e^{-3t}\]
(a) Using the result for $\sin kx$, and taking $k = 2$, we see that when $y = \cos \frac{x}{2}$, $\frac{dy}{dx} = 2 \cos 2x$.
\[- \frac{1}{2} \sin \frac{x}{2}\]

Back to the theory
5 sec^2 5x
1. (a) $18x$ (b) 0 (c) $18x^2$ (d) $-52x^3$ (e) $\frac{1}{7}$

2. (a) $-15t^{-4}$ (b) $\frac{3}{2}t^{\frac{1}{2}}$ (c) $-10t^{-3}$ (d) $-\frac{9}{4}t^{\frac{3}{2}}$

3. (a) $5\cos 5x$ (b) $-4\sin 4t$ (c) $3\sec^2 3r$ (d) $2e^{2x}$ (e) $-3e^{-3t}$

4. (a) $-\frac{2}{3}\sin \frac{2x}{3}$ (b) $-2\cos(-2x)$ (c) $\pi \sec^2 \pi x$ (d) $\frac{1}{2}e^{\frac{x}{2}}$ (e) $\frac{1}{x}$

Back to the theory.
$35x^4$
\[ 5e^{2x} \]
\[ 3(5e^{5x}) = 15e^{5x} \]
\[35x^4 - 15e^{5x}\]
\[-\frac{1}{2} \sin \frac{x}{2}\]
Back to the theory
\[-2 \sin \frac{x}{2} - 27x^2\]

Back to the theory
1. (a) $21x^6 + 24x^2$  (b) $-12x^3 + 3x^{0.5}$  (c) $-\frac{18}{x^3} - \frac{14}{x^2} - 3$
   (d) $\frac{1}{2}$  (e) $12 + 18x$

2. (a) $5 \cos t + 5 \cos 5t$  (b) $-6 \sin 2v - 3 \cos \frac{v}{2}$  (c) $8e^{2n} - 4e^{-2n} + n$
   (d) $\frac{3e^{3t}}{2} + 4 \sec^2 2t$  (e) $4r^3 + 4r + 8e^{-2r}$

3. (a) $6e^t + 2e^{2t}$  (b) $2\pi e^{2s} - \frac{1}{s^2} + 2\pi \cos(\pi s)$
   (c) $-\frac{2}{r^2} - \frac{2}{r} + 2r + 2$  (d) $12 \cos 2\theta + \frac{1}{2} \sin \frac{\theta}{4} + 4\theta$
   (e) $12 \sec^2 3t + 6 \cos 2t + 8 \sin 4t$

Back to the theory.
1. 1.380 2 (a) -1 (b) 5.8989 3 (a) 5 (b) 4.4305

Back to the theory