Introduction

The conic sections (or conics) - the ellipse, the parabola and the hyperbola - play an important role both in mathematics and in the application of mathematics to engineering. In this block we look in detail at the equations of the conics in both standard form and general form. Although there are various ways that can be used to define a conic we concentrate, in this block on defining conics using Cartesian coordinates \((x, y)\). However, at the end of this block we examine an alternative way to obtain the conics.

Prerequisites

Before starting this Block you should...

- thoroughly understand the various techniques of differentiation

Learning Outcomes

After completing this Block you should be able to...

- appreciate how conics are obtained as curves of intersection of a double-cone with a plane
- state the standard form of the equations of ellipse, the parabola and hyperbola
- classify quadratic expressions in \(x, y\) in terms of conics

Learning Style

To achieve what is expected of you...

- allocate sufficient study time
- briefly revise the prerequisite material
- attempt every guided exercise and most of the other exercises
1. The Ellipse, Parabola and Hyperbola

Mathematicians, engineers and scientists encounter numerous functions in their work: polynomials, trigonometric and hyperbolic functions amongst them. However, throughout the history of science one group of functions, the conics, arise time and time again not only in the development of mathematical theory but also in practical applications. The conics were first studied by the Greek mathematician Apollonius more than 200 years BC. Essentially, the conics form that class of curves which are obtained when a double cone is intersected by a plane. There are three main types: the ellipse, the parabola and the hyperbola. From the ellipse we obtain the circle as a special case, and from the hyperbola we obtain the rectangular hyperbola as a special case. These curves are illustrated in the following figures.

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**Circle:** obtained by intersection of a plane perpendicular to the cone-axis with cone

As the plane of intersection rotates the other conics are obtained

**Ellipse:** obtained by a plane, which is not perpendicular to the cone-axis, but cutting the cone in a closed curve. Various ellipses are obtained as the plane continues to rotate.

**Parabola:** obtained when the plane is parallel to the generator of the cone. Different parabolas are obtained as the point \( P \) moves along a generator.
The Ellipse

We are all aware that the paths followed by the planets around the sun are elliptical. However, more generally the ellipse occurs in many areas of engineering. The standard form of an ellipse is shown in Figure 1.

If \( a > b \) (as we have drawn the figure) then the \( x \)-axis is called the major-axis and the \( y \)-axis is called the minor-axis. On the other hand if \( b > a \) then the \( y \)-axis is called the major-axis and the \( x \)-axis is then the minor-axis. Two points, inside the ellipse are of importance; these are the foci. If \( a > b \) these are located at coordinate positions \( \pm ae \) (or at \( \pm be \) if \( b > a \)) on the major-axis in which \( e \), called the eccentricity, is given by

\[
e^2 = 1 - \frac{b^2}{a^2} \quad (b < a) \quad \text{or by} \quad e^2 = 1 - \frac{a^2}{b^2} \quad (a < b)
\]

The foci of an ellipse have the property that if light rays are emitted from one focus then on reflection at the elliptic curve they pass through at the other focus.
Key Point

The standard Cartesian equation of the ellipse with its centre at the origin is
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

This ellipse has intercepts on the x-axis at \( x = \pm a \) and on the y-axis at \( \pm b \). The curve is also symmetrical about both axes. The curve reduces to a circle in the special case in which \( a = b \).

Example

(a) Sketch the ellipse \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \)
(b) Find the eccentricity \( e \)
(c) Locate the positions of the foci.

Solution

(a) We can calculate the values of \( y \) as \( x \) changes from 0 to 2

\[
\begin{array}{cccccccc}
  x & 0 & 0.30 & 0.60 & 0.90 & 1.20 & 1.50 & 1.80 & 2 \\
  y & 3 & 2.97 & 2.86 & 2.68 & 2.40 & 1.98 & 1.31 & 0 \\
\end{array}
\]

From this table of values, and using the symmetry of the curve, a sketch can be drawn.

Here \( b = 3 \) and \( a = 2 \) so that the y-axis is the major axis and the x-axis is the minor axis. (b) \( e^2 = 1 - \frac{a^2}{b^2} = 1 - \frac{4}{9} = \frac{5}{9} \) \( \therefore e = \frac{\sqrt{5}}{3} \)

(c) The foci are located at \( \pm \sqrt{5} \) on the y-axis.

The Key Point above gives the equation of the ellipse with its centre at the origin. If the centre of the ellipse has coordinates \( (\alpha, \beta) \) and still has its axes parallel to the \( x \) and \( y \)-axes the standard equation becomes
\[ \frac{(x - \alpha)^2}{a^2} + \frac{(y - \beta)^2}{b^2} = 1. \]
The Circle

The circle is a special case of the ellipse; it occurs when \(a = b = r\) so the equation becomes

\[
\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1 \quad \text{or, more commonly} \quad x^2 + y^2 = r^2
\]

Here, the centre of the circle is located at the origin \((0, 0)\) and the radius of the circle is \(r\). If the centre of the circle at a point \((\alpha, \beta)\) then we use the form:

\[
(x - \alpha)^2 + (y - \beta)^2 = r^2
\]

**Key Point**

The equation of a circle with centre at \((\alpha, \beta)\) and radius \(r\) is

\[
(x - \alpha)^2 + (y - \beta)^2 = r^2
\]

In Figure 2 we have drawn a number of circles

![Figure 2](image-url)

The reader should confirm that the equations of these circles are

<table>
<thead>
<tr>
<th>circle</th>
<th>equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>((x - 1)^2 + (y - 1)^2 = 1)</td>
</tr>
<tr>
<td>B</td>
<td>((x - 3)^2 + (y - 1)^2 = 1)</td>
</tr>
<tr>
<td>C</td>
<td>((x + 0.5)^2 + (y + 2)^2 = 1)</td>
</tr>
<tr>
<td>D</td>
<td>((x - 2)^2 + (y + 2)^2 = 0.25)</td>
</tr>
<tr>
<td>E</td>
<td>((x + 0.5)^2 + (y - 2.5)^2 = 1)</td>
</tr>
</tbody>
</table>
Example Show that the expression

\[ x^2 + y^2 - 2x + 6y + 6 = 0 \]

represents the equation of a circle. Find its centre and radius.

Solution

We shall see later how to recognise this as the equation of a circle simply by examination of the coefficients of quadratic terms \( x^2, y^2 \). However, in the present example we will use the process of completing the square, for \( x \) and for \( y \), to show that the expression can be written in standard form. Now \( x^2 + y^2 - 2x + 6y + 6 \equiv x^2 - 2x + y^2 + 6y + 6. \) Also,

\[
\begin{align*}
  x^2 - 2x &\equiv (x - 1)^2 - 1 & \text{and} & \quad y^2 + 6y &\equiv (y + 3)^2 - 9.
\end{align*}
\]

Hence we can write

\[
\begin{align*}
  x^2 + y^2 - 2x + 6y + 6 &\equiv (x - 1)^2 - 1 + (y + 3)^2 - 9 + 6 = 0
\end{align*}
\]

or, taking the free constants to the right-hand side:

\[
(x - 1)^2 + (y + 3)^2 = 4.
\]

By comparing this with the standard form we conclude this represents the equation of a circle with centre at coordinate position \((1, -3)\) and radius 2.

Now do this exercise

Find the centre and radius of each of the following circles:

(a) \( x^2 + y^2 - 4x - 6y = -12 \)  
(b) \( 2x^2 + 2y^2 + 4x + 1 = 0 \)

Answer

The Parabola

The standard form of the parabola is shown in Figure 3.

Here the \( x \)-axis is the line of symmetry of the parabola.
Key Point

The standard equation of the parabola is

\[ y^2 = 4ax \]

with focus at \((a, 0)\).

It can be shown that light rays parallel to the \(x\)-axis will, on reflection from the parabolic curve, come together at the focus. This is an important property and is used in the construction of some kinds of telescopes, of satellite dishes and of car headlights.

Now do this exercise

Sketch the curve \(y^2 = 8x\). Find the position of the focus and confirm its light-focusing property.

By changing the equation of the parabola slightly we can change the position of the parabola along the \(x\)-axis. See Figure 4.

![Figure 4. Parabola \(y = 4a(x - b)\) has its vertex at \(x = b\).](image)

We can also have parabolae where the \(y\)-axis is the line of symmetry (see Figure 5). In this case the standard equation would be

\[ x^2 = 4ay \quad \text{or} \quad y = \frac{x^2}{4a} \]

![Figure 5.](image)

Now do this exercise

Sketch the curves \(y^2 = x\) and \(x^2 = 2(y - 3)\)

The focus of the parabola \(y^2 = 4a(x - b)\) is located at coordinate position \((a + b, 0)\). Also, changing the value of \(a\) changes the convexity of the parabola (see Figure 6).
The Hyperbola

The standard form of the hyperbola is shown in Figure 7(a).

![Figure 6.](image)

This has standard equation

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1
\]

The eccentricity, \(e\), is defined by

\[
e^2 = 1 + \frac{b^2}{a^2} \quad (e > 1)
\]

Note the change in sign when compared to the equivalent expressions for the ellipse. The lines \(y = \pm \frac{b}{a}x\) are asymptotes to the hyperbola (these are the lines to which each branch of the hyperbola approach as \(x \to \pm \infty\)).

If light is emitted from one focus then on hitting the hyperbolic curve it is reflected in such a way as to appear to be coming from the other focus. See Figure 7(b). The hyperbola has fewer uses in applications than the other conic sections and so we will not dwell here on its properties.
General Conics

The conics we have considered above - the ellipse, the parabola and the hyperbola - have all been presented in standard form:- their axes are parallel to either the x or y–axes. However, conics may be rotated to any angle with respect to the axes: they clearly remain conics, but what equations do they have?

It can be shown that the equation of any conic, can be described by the quadratic expression

\[ Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \]

where \( A, B, C, D, E, F \) are constants.

If not all of \( A, B, C \) are zero the graph of this equation is

(i) an ellipse if \( B^2 < 4AC \)  (circle if \( A = C \))
(ii) a parabola if \( B^2 = 4AC \)
(iii) a hyperbola if \( B^2 > 4AC \)

Example

Classify each of the following expressions as ellipse, parabola or hyperbola;

(a) \( x^2 + 2xy + 3y^2 + x - 1 = 0 \)
(b) \( x^2 + 2xy + y^2 - 3y + 7 = 0 \)
(c) \( 2x^2 + xy + 2y^2 - 2x + 3y = 6 \)

Solution

(a) Here \( A = 1, \ B = 2, \ C = 3 \)  \quad \therefore \quad B^2 < 4AC. \text{ This is an ellipse} \\
(b) Here \( A = 1, \ B = 2, \ C = 1 \)  \quad \therefore \quad B^2 = 4AC. \text{ This is a parabola} \\
(c) Here \( A = 2, \ B = 1, \ C = 2 \)  \quad \therefore \quad B^2 < 4AC \text{ also } A = C. \text{ This is a circle.} \\

Now do this exercise

Classify each of the following conics:

(a) \( x^2 - 2xy - 3y^2 + x - 1 = 0 \)
(b) \( 2x^2 + xy - y^2 - 2x + 3y = 0 \)
(c) \( 4x^2 - y + 3 = 0 \)
(d) \( -x^2 - xy - y^2 + 3x = 0 \)
More exercises for you to try

1. The equation $9x^2 + 4y^2 - 36x + 24y - 1 = 0$ represents an ellipse. Find its centre, the semi-major and semi-minor axes and the coordinate positions of the foci.

2. Find the equation of a circle of radius 3 which has its centre at $(-1, 2.2)$

3. Find the centre and radius of the circle $x^2 + y^2 - 2x - 2y - 5 = 0$

4. Find the position of the focus of the parabola $y^2 - x + 3 = 0$

5. Classify each of the following conics

   (a) $x^2 + 2x - y - 3 = 0$
   (b) $8x^2 + 12xy + 17y^2 - 20 = 0$
   (c) $x^2 + xy - 1 = 0$
   (d) $4x^2 - y^2 - 4y = 0$
   (e) $6x^2 + 9y^2 - 24x - 54y + 51 = 0$

6. An asteroid has an elliptical orbit around the Sun. The major axis is of length $5 \times 10^8$ km. If the distance between the foci is $4 \times 10^8$ km find the equation of the orbit.

Answer
2. Computer Exercise or Activity

For this exercise it will be necessary for you to access the computer package DERIVE.

DERIVE will not plot curves if their equation is expressed in *implicit Cartesian form*. The curve will need to be expressed in *explicit Cartesian form*. As an example \( x^2 + y^2 = 1 \) is in implicit Cartesian form. We can solve for \( y \) to give \( y = \pm \sqrt{1-x^2} \) which is in explicit Cartesian form. However, DERIVE will still not plot this function as to each value of \( x \) there corresponds two values of \( y \). DERIVE will plot \( y = \sqrt{1-x^2} \) and \( y = -\sqrt{1-x^2} \) separately. Each gives one half of a circle.

We will find it easier to plot conics when we examine either the polar form of a conic or the parametric representation of a conic in the next two blocks.

Your plot of the circle may appear elliptical. To get the appearance of a circle you will need to choose appropriate values for **Horizontal** and **Vertical** intervals in the **Options: Grids** dialog box in the **Windows** page.

If you wish to plot conic sections in which the equations are written in implicit Cartesian form then proceed as in the following example.

Consider the ellipse \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \).

In DERIVE, 

**Author:** Expression \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \) then **Solve:** Algebraically (choose **Variable** \( y \)) followed by **Simplify**.

**DERIVE** responds

\[
\begin{align*}
#3 & \left[ y = \frac{3\sqrt{4-x^2}}{2}, \quad y = -\frac{3\sqrt{4-x^2}}{2} \right] \\
\end{align*}
\]

Now plot each of the curves \( y = \frac{3\sqrt{4-x^2}}{2}, \ y = -\frac{3\sqrt{4-x^2}}{2} \) separately to give the full ellipse. (Copy #3 as an expression within the **Author: Expression** dialog box and delete various parts as appropriate).
End of Block 17.1
(a) centre: (2, 3) radius 1  
(b) centre: (−1, 0) radius $\sqrt{2}/2$. 

Back to the theory
This is a standard parabola \((y^2 = 4ax)\) with \(a = 2\). Thus the focus is located at coordinate position \((2,0)\).

If your sketch is sufficiently accurate you should find that light-rays (lines) parallel to the \(x\)-axis when reflected off the parabolic surface pass through the focus. (Draw a tangent at the point of reflection and ensure that the angle of incidence (\(\theta\) say) is the same as the angle of reflection.)
Back to the theory.
(a) \( A = 1, B = -2, C = -3 \) \( B^2 > 4AC \) \( \therefore \) hyperbola

(b) \( A = 2, B = 1, C = -1 \) \( B^2 > 4AC \) \( \therefore \) hyperbola

(c) \( A = 4, B = 0, C = 0 \) \( B^2 = 4AC \) \( \therefore \) parabola

(d) \( A = -1, B = -1, C = -1 \) \( B^2 < 4AC \), \( A = C \) \( \therefore \) circle
1. centre: (2, −3), semi-major 3, semi-minor 2, foci: (2, −3 ± √5)

2. \((x + 1)^2 + (y − 2.2)^2 = 9\)

3. centre: (1, 1) radius \(\sqrt{7}\)

4. \(y^2 = (x - 3) \therefore a = 1, b = -3\). Hence focus is at coordinate position (4, 0).

5. (a) parabola with vertex (−1, −4)
   (b) ellipse
   (c) hyperbola
   (d) hyperbola
   (e) ellipse with centre (2,3)

6. \(9x^2 + 25y^2 = 5.625 \times 10^7\)

Back to the theory