Introduction

Blocks 19.2 to 19.6 have introduced several techniques for solving commonly-occurring first-order and second-order ordinary differential equations. In this Block we solve a number of these equations which model engineering systems.

Prerequisites

Before starting this Block you should ...

① understand what is meant by a differential equation; (Block 19.1)
② be familiar with the terminology associated with differential equations: order, dependent variable and independent variable; (Block 19.1)
③ be able to integrate; (Blocks 14.1-14.8)
④ have completed Blocks 19.2, 19.4, 19.5 and 19.6

Learning Outcomes

After completing this Block you should be able to ...

✓ recognise and solve first-order ordinary differential equations, modelling simple electrical circuits, projectile motion and Newton’s law of cooling
✓ recognise and solve second-order ordinary differential equations with constant coefficients modelling free electrical and mechanical oscillations
✓ recognise and solve second-order ordinary differential equations with constant coefficients modelling forced electrical and mechanical oscillations

Learning Style

To achieve what is expected of you ...

分配 sufficient study time
briefly revise the prerequisite material
attempt every guided exercise and most of the other exercises
1. Modelling with First-order Equations

**Applying Newton’s law of cooling**

In Block 19.1 we introduced Newton’s law of cooling. The model equation was

\[
\frac{d\theta}{dt} = -k(\theta - \theta_s)
\]  

(1)

where \( \theta = \theta(t) \) is the temperature of the cooling object at time \( t \), \( \theta_s \) the temperature of the environment (assumed constant) and \( k \) is a thermal constant related to the object. Let \( \theta_0 \) be the initial temperature of the liquid, i.e.

\[
\theta = \theta_0 \quad \text{at} \quad t = 0.
\]

**Try each part of this exercise**

Solve this initial value problem.

Part (a) Separate the variables to obtain an equation connecting two integrals

Answer

Part (a) Now integrate both sides of this equation

Answer

Part (a) Apply the initial condition and take exponentials to obtain a formula for \( \theta \)

Answer

Hence \( \ln(\theta - \theta_s) = -kt + \ln(\theta_0 - \theta_s) \) so that \( \ln(\theta - \theta_s) - \ln(\theta_0 - \theta_0) = -kt \)

Thus, rearranging and inverting, we find:

\[
\ln\left(\frac{\theta - \theta_s}{\theta_0 - \theta_s}\right) = -kt \quad \therefore \quad \frac{\theta - \theta_s}{\theta_0 - \theta_s} = e^{-kt}
\]

and so, finally, \( \theta = \theta_s + (\theta_0 - \theta_s)e^{-kt} \).

The graph of \( \theta \) against \( t \) is shown in the figure below.

![Graph of \( \theta \) against \( t \)](image)

We see that as time increases \( (t \to \infty) \), then the temperature of the object cools down to that of the environment, that is: \( \theta \to \theta_s \).

Note that we could have solved (1) by the integrating factor method.
Try each part of this exercise

Part (a) Write the equation as
\[ \frac{d\theta}{dt} + k\theta = k\theta_s \quad (2) \]
What is the integrating factor for this equation?

Multiplying (2) by this factor we find that
\[ e^{kt} \frac{d\theta}{dt} + k e^{kt} \theta = k\theta_s e^{kt} \quad \text{or, rearranging,} \quad \frac{d}{dt} (e^{kt} \theta) = k\theta_s e^{kt} \]

Part (b) Now integrate this equation and apply the initial condition

Hence \( \theta = \theta_s + C e^{-kt} \). Then, applying the initial condition: when \( t = 0 \), \( \theta = \theta_s + C \) so that \( C = \theta_0 - \theta_s \) and, finally,
\[ \theta = \theta_s + (\theta_0 - \theta_s) e^{-kt}, \]
as before.

Electrical circuits

Another application of first-order differential equations arises in the modelling of electrical circuits.

In Block 19.1 the differential equation for the RL circuit of the figure below was shown to be
\[ L \frac{di}{dt} + Ri = E \]
in which the initial condition is \( i = 0 \) at \( t = 0 \).

Try each part of this exercise

Part (a) Write this equation in standard form \( \{ \frac{dy}{dx} + P(x)y = Q(x) \} \) and obtain the integrating factor.

Multiplying the equation in standard form by the integrating factor gives
\[ e^{Rt/L} \frac{di}{dt} + e^{Rt/L} \frac{R}{L} i = \frac{E}{L} e^{Rt/L} \]
or, rearranging,
\[ \frac{d}{dt} \left( e^{Rt/L} i \right) = \frac{E}{L} e^{Rt/L}. \]
Part (b) Now integrate both sides and apply the initial condition to obtain the solution

2. Modelling Free Mechanical Oscillations

Consider the following schematic diagram of a shock absorber:

![Schematic of a shock absorber](image)

The equation of motion can be described in terms of the vertical displacement $x$ of the mass. Let $m$ be the mass, $k \frac{dx}{dt}$ the damping force resulting from the dashpot and $nx$ the restoring force resulting from the spring. Here, $k$ and $n$ are constants. Then the equation of motion is

$$m \frac{d^2x}{dt^2} = -k \frac{dx}{dt} - nx.$$

Suppose that the mass is displaced a distance $x_0$ initially and released from rest. Then at $t = 0$, $x = x_0$ and $\frac{dx}{dt} = 0$. Writing the differential equation in standard form:

$$m \frac{d^2x}{dt^2} + k \frac{dx}{dt} + nx = 0.$$

We shall see that the nature of the oscillations described by this differential equation, depends crucially upon the relative values of the mechanical constants $m, k$ and $n$.

**Now do this exercise**

Find the auxiliary equation of this differential equation and solve it.

The value of $k$ controls the amount of damping in the system. We explore the solution for two particular values of $k$. 
No damping

If \( k = 0 \) then there is no damping. We expect, in this case, that once motion has started it will continue for ever. The motion that ensues is called simple harmonic motion. In this case we have

\[
\lambda = \frac{\pm \sqrt{-4mn}}{2m},
\]

that is,

\[
\lambda = \pm \sqrt{\frac{n}{m}} i \quad \text{where} \quad i = \sqrt{-1}.
\]

In this case the solution for the displacement \( x \) is:

\[
x = A \cos \left( \sqrt{\frac{n}{m}} t \right) + B \sin \left( \sqrt{\frac{n}{m}} t \right)
\]

where \( A, B \) are arbitrary constants.

Now do this exercise

Now apply the initial conditions to find the unique solution:

Answer

Light damping

If \( k^2 - 4mn < 0 \), i.e. \( k^2 < 4mn \) then the roots of the auxiliary equation are complex:

\[
\lambda_1 = \frac{-k + i\sqrt{4mn - k^2}}{2m} \quad \lambda_2 = \frac{-k - i\sqrt{4mn - k^2}}{2m}
\]

Then, after some rearrangement:

\[
x = e^{-kt/2m} [A \cos pt + B \sin pt]
\]

in which \( p = \sqrt{4mn - k^2}/2m \).

Try each part of this exercise

Part (a) If \( m = 1, \ n = 1 \) and \( k = 1 \) find \( \lambda_1 \) and \( \lambda_2 \) and then find the solution for the displacement \( x \).

Answer

Part (b) Impose the initial conditions \( x = x_0, \ \frac{dx}{dt} = 0 \) at \( t = 0 \) to find the arbitrary constants.

Answer
The graph of $x$ against $t$ is shown below. This is the case of light damping. As the damping in the system decreases (i.e. $k \to 0$) the number of oscillations (in a given time interval) will increase. In many mechanical systems these oscillations are usually unwanted and the designer would choose a value of $k$ to either reduce them or to eliminate them altogether. For the choice $k = 4mn$, known as the critical damping case, all the oscillations are absent.

\[
x = x_0 e^{-t/2} \left[ \cos \frac{\sqrt{3}}{2} t + \frac{\sqrt{3}}{3} \sin \frac{\sqrt{3}}{2} t \right]
\]

**Heavy damping**

If $k^2 - 4mn > 0$, i.e. $k^2 > 4mn$ then there are two real roots of the auxiliary equation, $\lambda_1$ and $\lambda_2$:

\[
\lambda_1 = \frac{-k + \sqrt{k^2 - 4mn}}{2m} \quad \lambda_2 = \frac{-k - \sqrt{k^2 - 4mn}}{2m}
\]

Then

\[
x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}.
\]

**Try each part of this exercise**

Part (a) If $m = 1$, $n = 1$ and $k = 2.5$ find $\lambda_1$ and $\lambda_2$ and then find the solution for the displacement $x$.

Part (b) Impose the initial conditions $x = x_0$, $\frac{dx}{dt} = 0$ at $t = 0$ to find the arbitrary constants.

3. Modelling Forced Mechanical Oscillations

Suppose now that the mass is subject to a force $f(t)$ after the initial disturbance. Then the equation of motion is

\[
m \frac{d^2x}{dt^2} + k \frac{dx}{dt} + nx = f(t)
\]

Consider the case $f(t) = F \cos \omega t$, that is, an oscillatory force of magnitude $F$ and angular frequency $\omega$. Choosing specific values for the constants in the model: $m = n = 1$, $k = 0$, and $\omega = 2$ we find

\[
\frac{d^2x}{dt^2} + x = F \cos 2t
\]
Try each part of this exercise

Part (a) Find the complementary function for this equation.

Part (b) Now find a particular integral for the differential equation.

Part (c) Finally, apply the initial conditions to find the solution for the displacement $x$.

If the angular frequency $\omega$ of the applied force is nearly equal to that of the free oscillation the phenomenon of beats occurs. If the angular frequencies are equal we get the phenomenon of resonance. Both these phenomena are dealt with in the Computer Exercises. Note that we can eliminate resonance by introducing damping into the system. See the Computer Exercises.
More exercises for you to try

1. In an RC circuit (a resistor and a capacitor in series) the applied emf is a constant $E$. Given that $\frac{dq}{dt} = i$ where $q$ is the charge in the capacitor, $i$ the current in the circuit, $R$ the resistance and $C$ the capacitance the equation for the circuit is

$$Ri + \frac{q}{C} = E.$$ 

If the initial charge is zero find the charge subsequently.

2. If the voltage in the RC circuit is $E = E_0 \cos \omega t$ find the charge and the current at time $t$.

3. An object is projected from the Earth’s surface. What is the least velocity (the escape velocity) of projection in order to escape the gravitational field, ignoring air resistance. The equation of motion is

$$m v \frac{dv}{dx} = -m g \frac{R^2}{x^2}$$

where the mass of the object is $m$, its distance from the centre of the Earth is $x$ and the radius of the Earth is $R$.

4. The radial stress $p$ at distance $r$ for the axis of the thick cylinder subjected to internal pressure is given by $p + r \frac{dp}{dr} = A - p$ where $A$ is a constant. If $p = p_0$ at the inner wall $r = r_1$ and is negligible ($p = 0$) at the outer wall $r = r_2$ find an expression for $p$.

5. The equation for an LCR circuit with applied voltage $E$ is

$$L \frac{di}{dt} + Ri + \frac{1}{C}q = E.$$ 

By differentiating this equation find the solution for $q(t)$ and $i(t)$ if $L = 1$, $R = 100$, $C = 10^{-4}$ and $E = 1000$ given that $q = 0$ and $i = 0$ at $t = 0$.

6. Consider the free vibration problem when $m = 1$, $n = 1$ and $k = 2$. (Critical damping) Find the solution for $x(t)$.

7. Repeat problem 6 for the case $m = 1$, $n = 1$ and $k = 1.5$ (light damping)

8. Consider the forced vibration problem with $m = 1$, $n = 25$, $k = 8$, $E = \sin 3t$, $x_0 = 0$ with an initial velocity of 3.

Answer
4. Computer Exercise or Activity

For this exercise it will be necessary for you to access the computer package DERIVE.

To solve a differential equation using DERIVE it is necessary to load what is called a Utility File. In this case you will either need to load ode1 or ode2. To do this is simple. Proceed as follows: In DERIVE, choose File:Load:Math and select the file (double click) on the ode1 or ode2 icon. This will load a number of commands which enable you to solve first-order and second-order differential equations. You can use the Help facility to learn more about these if you wish.

Also note that many of the differential equations presented in this Block are linear differential equations having the general form

$$\frac{dy}{dx} + p(x)y = q(x) \quad y(x_0) = y_0$$

Such equations can also be solved in DERIVE using the command Linear1(p,q,x,y,x0,y0) or by using the related command Linear1.Gen(p,q,x,y,c) for a solution to a differential equation without initial conditions but which contains a single arbitrary constant c.

Also use the Help command to find out about the more general commands Dsolve1(p, q, x, y, x0, y0) and Dsolve1.Gen(p, q, x, y, c) used for solving very general first-order ordinary differential equations.

For second-order equations use the DERIVE command

$$\text{Dsolve2}(p, q, r, x, c1, c2)$$

which finds the general solution (containing two arbitrary constants c1, c2) to the second order differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

For many of the examples in this Block both p(x) and q(x) are given constants.

As an exercise use DERIVE to check the correctness of the solutions requested in the examples and guided exercises of this Block.

As a more involved exercise use DERIVE to explore the solution to

$$m\frac{d^2x}{dt^2} + k \frac{dx}{dt} + nx = f(t)$$

with

$$k^2 < 4mn$$

(trigonometric solutions with frequency \(\omega_s = \sqrt{\frac{4mn - k^2}{2m}}\)) and when

$$f(t) = \sin \omega t.$$ You will find that after an initial period the system will vibrate with the same frequency as the forcing function. The initial response is affected by the transient function (the complementary function). If there is damping in the system this will die away, due to the decaying exponential terms in the complementary function, to leave only that part of the solution arising from the particular integral. An interesting case arises when \(\omega \approx \omega_s\). You will find, by plotting the solution curve, that the phenomenon of \textbf{beats} occurs. That is, the system ‘beats’ with a frequency neither of the system frequency nor of the frequency of the forcing function. As \(\omega \to \omega_s\) the magnitude of the peaks and troughs of the solution curve increase.
If there is no damping in the system the response becomes unbounded as time increases in the limit $\omega = \omega_s$. The system is in resonance with the forcing function. These extreme oscillations can be reduced by introducing damping into the system. Examine this phenomenon as the damping ($k$) is reduced.

MAPLE will solve a wide range of ordinary differential equations including systems of differential equations using the command

$$\text{dsolve}(\text{deqns}, \text{vars}, \text{eqns})$$

where:
- **deqns** – ordinary differential equation in vars, or set of equations and/or initial conditions.
- **vars** – variable or set of variables to be solved for
- **eqns** – optional equation of the form keyword=value

For example to solve

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = e^{-t} \quad y(0) = 0, \quad y'(0) = 0$$

we would key in

```maple
> dsolve({diff(y(t),t$2)+2*diff(y(t),t)+2*y(t)=exp(-t),y(0)=0,D(y)(0)=0},y(t),type=exact);
```

MAPLE responds with

$$\frac{1 - \cos(t)}{e^t}$$

If the initial conditions are omitted MAPLE will present the solution with the correct number of arbitrary constants denoted by $C_1, C_2, \ldots$. Thus the general solution of

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = e^{-t}$$

is obtained by keying in

```maple
> dsolve({diff(y(t),t$2)+2*diff(y(t),t)+2*y(t)=exp(-t)},{y(t),type=exact});
```

and MAPLE responds with

$$y(t) = \exp(-t) + C_1 \exp(-t) \cos(t) + C_2 \exp(-t) \sin(t)$$
\[
\int \frac{d\theta}{\theta - \theta_a} = - \int k \, dt
\]
\[ \ln(\theta - \theta_s) = -kt + C \text{ where } C \text{ is constant} \]
\ln(\theta_0 - \theta_s) = C.
$e^\int k\,dt = e^{kt}$ is the integrating factor.

Back to the theory
\[ \theta = \theta_s + (\theta_0 - \theta_s)e^{-kt} \text{ since integration produces } e^{kt}\theta = \theta_se^{kt} + C, \text{ where } C \text{ is an arbitrary constant.} \]
divide the differential equation through by $L$ to obtain

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$$

which is now in standard form. The integrating factor is $e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$.
You should obtain $i = \frac{E}{R} (1 - e^{-Rt/L})$, since integrating the differential equation gives:

$$e^{Rt/L} i = \frac{E}{R} e^{Rt/L} + C$$

where $C$ is a constant. Then applying the initial condition $i = 0$ when $t = 0$ gives

$$0 = \frac{E}{R} + C$$

so that $C = -\frac{E}{R}$ and $e^{Rt/L} i = \frac{E}{R} e^{Rt/L} - \frac{E}{R}$. Finally, $i = \frac{E}{R} (1 - e^{-Rt/L})$. Note that as $t \to \infty$, $i \to \frac{E}{R}$.
Put $x = e^{\lambda t}$ then the auxiliary equation is

$$m \lambda^2 + k \lambda + n = 0.$$

Hence $\lambda = \frac{-k \pm \sqrt{k^2 - 4mn}}{2m}$.

Back to the theory.
You should obtain \( x = x_0 \cos \left( \sqrt{\frac{n}{m}} t \right) \). To see this we first need an expression for the derivative, \( \frac{dx}{dt} \):

\[
\frac{dx}{dt} = -\sqrt{\frac{n}{m}} A \sin \left( \sqrt{\frac{n}{m}} t \right) + \sqrt{\frac{n}{m}} B \cos \left( \sqrt{\frac{n}{m}} t \right)
\]

When \( t = 0 \), \( \frac{dx}{dt} = 0 \) so that

\[
\sqrt{\frac{n}{m}} B = 0 \quad \text{so that} \quad B = 0.
\]

Therefore

\[
x = A \cos \left( \sqrt{\frac{n}{m}} t \right).
\]

Imposing the remaining initial condition: when \( t = 0 \), \( x = x_0 \) so that \( x_0 = A \) and finally:

\[
x = x_0 \cos \left( \sqrt{\frac{n}{m}} t \right).
\]
\[ \lambda = \frac{-1 + i \sqrt{4 - 1}}{2} = -1/2 \pm i \sqrt{3}/2. \text{ Hence } x = e^{-t/2} \left[ A \cos \frac{\sqrt{3}}{2} t + B \sin \frac{\sqrt{3}}{2} t \right]. \]
Differentiating, we obtain
\[
\frac{dx}{dt} = -\frac{1}{2}e^{-t/2} \left[ A \cos \frac{\sqrt{3}}{2} t + B \sin \frac{\sqrt{3}}{2} t \right] + e^{-t/2} \left[ -\frac{\sqrt{3}}{2} A \sin \frac{\sqrt{3}}{2} t + \frac{\sqrt{3}}{2} B \cos \frac{\sqrt{3}}{2} t \right]
\]

At \( t = 0, \)
\[
x = x_0 = A \quad \text{(i)}
\]
\[
\frac{dx}{dt} = 0 = -\frac{1}{2} A + \frac{\sqrt{3}}{2} B \quad \text{(ii)}
\]

Solving (i) and (ii) we obtain
\[
A = x_0 \quad B = \frac{\sqrt{3}}{3} x_0
\]
Then
\[
x = x_0 e^{-t/2} \left[ \cos \frac{\sqrt{3}}{2} t + \frac{\sqrt{3}}{3} \sin \frac{\sqrt{3}}{2} t \right].
\]

Back to the theory.
\[ \lambda = \frac{-2.5 \pm \sqrt{6.25 - 4}}{2} = -1.25 \pm 0.75 \]

Hence \( \lambda_1, \lambda_2 = -0.5, -2 \) and so \( x = Ae^{-0.5t} + Be^{-2t} \)

Back to the theory.
Differentiating, we obtain
$$\frac{dx}{dt} = -0.5Ae^{-0.5t} - 2Be^{-2t}$$

At $t = 0$,

$$x = x_0 = A + B \quad (i)$$
$$\frac{dx}{dt} = 0 = -0.5A - 2B \quad (ii)$$

Solving (i) and (ii) we obtain

$$A = \frac{4}{3}x_0 \quad B = -\frac{1}{3}x_0$$

Then

$$x = \frac{1}{3}x_0(4e^{-0.5t} - e^{-2t}).$$

The graph of $x$ against $t$ is shown below. This is the case of heavy damping. Other cases are dealt with in the Exercises at the end of the Block.
The homogeneous equation is
\[ \frac{d^2x}{dt^2} + x = 0 \]
with auxiliary equation \( \lambda^2 + 1 = 0 \). Hence the complementary function is
\[ x_{ct} = A \cos t + B \sin t. \]
Say
\[ x_p = C \cos 2t + D \sin 2t \]
so that
\[ \frac{d^2x_p}{dt^2} = -4C \cos 2t - 4D \sin 2t. \]
Substituting into the differential equation
\[ (-4C + C) \cos 2t + (-4D + D \sin 2t) \equiv F \cos 2t. \]
Comparing coefficients gives
\[ -3C = F \quad \text{and} \quad -3D = 0 \]
so that
\[ D = 0 \quad C = -\frac{1}{3}F \quad x_p = -\frac{1}{3}F \cos 2t. \]
The general solution of the differential equation is therefore
\[ x = x_p + x_c = -\frac{1}{3}F \cos 2t + A \cos t + B \sin t. \]
You should obtain

\[ x = -\frac{1}{3}F \cos 2t + (x_0 + \frac{1}{3}F) \cos t. \]

To obtain this we need to determine the derivative and apply the initial conditions:

\[ \frac{dx}{dt} = \frac{2}{3}F \sin 2t - A \sin t + B \cos t. \]

At \( t = 0 \)

\[ x = x_0 = -\frac{1}{3}F + A \quad (i) \]

\[ \frac{dx}{dt} = 0 = B \quad (ii) \]

Hence

\[ B = 0 \quad \text{and} \quad A = x_0 + \frac{1}{3}F. \]

Then

\[ x = -\frac{1}{3}F \cos 2t + (x_0 + \frac{1}{3}F) \cos t. \]

The graph of \( x \) against \( t \) is shown below.

[Graph showing the function \( x = -\frac{1}{3}F \cos 2t + (x_0 + \frac{1}{3}F) \cos t \) against \( t \).]

Back to the theory.
1. Use the equation in the form \( R \frac{dq}{dt} + \frac{q}{C} = E \) or \( \frac{dq}{dt} + \frac{1}{RC}q = \frac{E}{R} \). The integrating factor is \( e^{t/RC} \) and the general solution is 
\[ q = EC(1 - e^{-t/RC}) \] and as \( t \to \infty \) \( q \to EC \).

2. \( q = \frac{E_0C}{1 + \omega^2R^2C^2}[\cos \omega t - e^{-t/RC} + \omega RC \sin \omega t] \)
\[ i = \frac{dq}{dt} = \frac{E_0C}{1 + \omega^2R^2C^2}[\omega \sin \omega t + \frac{1}{RC} e^{-t/RC} + \omega^2 RC \cos \omega t]. \]

3. \( v_{\text{min}} = \sqrt{2gR} \). If \( R = 6378 \text{ km} \) and \( g = 9.81 \text{ m s}^{-2} \) then \( v_{\text{min}} = 11.2 \text{ km s}^{-1} \).

4. \( p = \frac{p_0r_1^2}{r_2^2 - r_1^2} \left( 1 - \frac{r_2}{r_1} \right) \)

5. \( q = 0.1 - \frac{1}{10\sqrt{3}}e^{-50t}(\sin 50\sqrt{3}t + \sqrt{3} \cos 50\sqrt{3}t) \) \( i = \frac{20}{\sqrt{3}}e^{-50t} \sin 50\sqrt{3}t \).

6. \( x = x_0(1 + t)e^{-t} \)

7. \( x = x_0e^{-0.75t}(\cos \frac{\sqrt{3}}{4}t + \frac{3}{\sqrt{3}} \sin \frac{\sqrt{3}}{4}t) \) \( 8. \)
\[ x = \frac{1}{104} \left[ e^{-4t} (3 \cos 3t + 106 \sin 3t) - 3 \cos 3t + 2 \sin 3t) \right] \]