Introduction

In this Block we employ the Laplace transform to solve constant coefficient ordinary differential equations. In particular we shall consider initial value problems. We shall find that the initial conditions are automatically included as part of the solution process. The idea is simple; the Laplace transform of each term in the differential equation is taken. If the unknown function is \( y(t) \) then, on taking the transform, an algebraic equation involving \( Y(s) = \mathcal{L}\{y(t)\} \) is obtained. This equation is solved for \( Y(s) \) which is then inverted to produce the required solution \( y(t) = \mathcal{L}^{-1}\{Y(s)\} \).

Prerequisites

Before starting this Block you should...

1. understand how to find Laplace transforms of simple functions and their derivatives
2. be able to find inverse Laplace transforms using a variety of techniques
3. understand what an initial-value problem is

Learning Outcomes

After completing this Block you should be able to...

✓ solve initial-value problems using the Laplace transform method

Learning Style

To achieve what is expected of you...

分配 sufficient study time

briefly revise the prerequisite material

attempt every guided exercise and most of the other exercises
1. Solving Differential Equations using the Laplace Transform

We begin with a straightforward initial value problem involving a first order constant coefficient differential equation. Let us find the solution of

$$\frac{dy}{dt} + 2y = 12e^{3t} \quad y(0) = 3$$

using the Laplace transform approach.

Although it is not stated explicitly we shall assume that \(y(t)\) is a causal function (we have no interest in the value of \(y(t)\) if \(t < 0\)). Similarly, the function on the right-hand side of the differential equation \((12e^{3t})\), the ‘forcing function’, will be assumed to be causal. (Strictly, we should write \(12e^{3t}u(t)\) but the step function \(u(t)\) will often be omitted). Let us write \(\mathcal{L}\{y(t)\} = Y(s)\). Then, taking the Laplace transform of every term in the differential equation gives:

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + \mathcal{L}\{2y\} = \mathcal{L}\{12e^{3t}\}$$

Now

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = -y(0) + sY(s) = -3 + sY(s)$$

$$\mathcal{L}\{2y\} = 2Y(s)$$

and

$$\mathcal{L}\{12e^{3t}\} = \frac{12}{s - 3}.$$ 

Substituting these expressions into the transformed version of the differential equation gives:

$$[-3 + sY(s)] + 2Y(s) = \frac{12}{s - 3}$$

Solving for \(Y(s)\) we have

$$(s + 2)Y(s) = \frac{12}{s - 3} + 3 = \frac{3 + 3s}{s - 3}$$

Therefore

$$Y(s) = \frac{3(s + 1)}{(s + 2)(s - 3)}.$$ 

Now, using partial fractions, this last expression can be written in a more convenient form:

$$Y(s) = \frac{3/5}{(s + 2)} + \frac{12/5}{(s - 3)}$$

and then, inverting:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s + 2}\right\} + \frac{12}{5}\mathcal{L}^{-1}\left\{\frac{1}{s - 3}\right\}$$

thus

$$y(t) = \frac{3}{5}e^{-2t}u(t) + \frac{12}{5}e^{3t}u(t)$$

This is the solution to the given initial value problem.
Try each part of this exercise

The equation governing the build up of charge, \( q(t) \), on the capacitor of an \( RC \) circuit is

\[
R \frac{dq}{dt} + \frac{1}{C} q = v_0
\]

where \( v_0 \) is the constant d.c. voltage. Initially, the circuit is relaxed and the circuit ‘closed’ at \( t = 0 \) and so \( q(0) = 0 \) is the initial condition for the charge.

Use the Laplace transform method to solve the differential equation for \( q(t) \). Assume the forcing term \( v_0 \) is causal.

Part (a) Begin by finding an expression for \( Q(s) = \mathcal{L}\{q(t)\} \)

Part (b) Now expand the expression using partial fractions

Part (c) Now obtain \( q(t) \) by taking inverse Laplace transforms

The Laplace transform method is also applied to higher-order differential equations in a similar way.

Example Solve the second-order initial-value problem:

\[
\frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + 2y = e^{-t} \quad y(0) = 0 \quad y'(0) = 0
\]

using the Laplace transform method.

Solution

As usual we shall assume the forcing function is causal (i.e. is really \( e^{-t} u(t) \)). Taking the Laplace transform of each term:

\[
\mathcal{L}\{\frac{d^2 y}{dt^2}\} + 2\mathcal{L}\{\frac{dy}{dt}\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}
\]

that is,

\[
[-y'(0) - sy(0) + s^2 Y(s)] + 2[-y(0) + sY(s)] + 2Y(s) = \frac{1}{s + 1}
\]

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20.4: The Laplace Transform
Solution
Inserting the initial conditions and rearranging:

\[ Y(s)[s^2 + 2s + 2] = \frac{1}{s+1} \]

i.e. \[ Y(s) = \frac{1}{(s+1)(s^2 + 2s + 2)} \]

Then, using partial fractions:

\[ \frac{1}{(s+1)(s^2 + 2s + 2)} = \frac{1}{s+1} - \frac{s+1}{(s+1)(s+1)^2 + 1} \]

where we have completed the square in the second term of the right-hand-side. We can now take the inverse Laplace transform:

\[ y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{1}{s+1}\} - \mathcal{L}^{-1}\{\frac{s+1}{(s+1)^2 + 1}\} \]

\[ = (e^{-t} - e^{-t} \cos t)u(t) \]

which is the solution to the initial value problem.

More exercises for you to try

Use Laplace transforms to solve:

1. \[ \frac{d^2x}{dt^2} + x = 2t \quad x(0) = 0 \quad x'(0) = 5 \]

2. \[ \frac{dx}{dt} + x = 9e^{2t} \quad x(0) = 3 \]
2. Computer Exercise or Activity

For this exercise it will be necessary for you to access the computer package DERIVE.

In the world of professional mathematics there are better ways of solving differential equations than using the Laplace transform. However, if you wish to check the solutions to the exercises above DERIVE can help. To solve a differential equation of first (second) order using DERIVE it is necessary to load what is called a Utility File named ode1 (or ode2). To do this is simple. Proceed as follows: In DERIVE, choose File:Load:Math and select the file (double click) on the ode1 (or ode2 as appropriate) icon. This will load a number of commands which enable you to solve first(second)-order differential equations. You can use the Help facility to learn more about these if you wish.

Of particular relevance are the DERIVE commands Dsolve1(\(p, q, x, y, x0, y0\)) which provides a solution to a first-order differential equation of the form

\[
\frac{dy}{dx} + p(x)y = q(x) \quad y(x0) = y0
\]

and the command Dsolve2.iv(\(p, q, r, x, y, x0, y0, v0\)) which provides a solution to the second-order initial value problem:

\[
\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x) \quad y(x0) = y0, \quad y'(x0) = v0
\]

MAPLE will solve a wide range of ordinary differential equations using the command
dsolve(\{deqns,vars,eqns\})

where:
- deqns — ordinary differential equation in vars, or set of equations and/or initial conditions.
- vars — variable or set of variables to be solved for
- eqns — optional equation of the form keyword=value

For example to solve

\[
\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = e^{-t} \quad y(0) = 0, \quad y'(0) = 0
\]

we would key in

> dsolve({diff(y(t),t$2)+2*diff(y(t),t)+2*y(t)=exp(-t),y(0)=0,D(y)(0)=0},y(t),type=exact);

MAPLE responds with \(\frac{1 - \cos(t)}{e^t}\).

If your differential equation involves either a step function or a delta function then you must use method=laplace optional equation as part of the eqns list. This ensures that the differential equation will be solved using Laplace transforms which is necessary as the normal dsolve commands do not recognise step or delta functions. For example to solve

\[
\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = e^{-t}u(t) \quad y(0) = 0, \quad y'(0) = 0
\]
we would key in
> dsolve(
{diff(y(t),t$2)+2*diff(y(t),t)+2*y(t)=exp(-t)*Heaviside(t), y(0)=0,D(y)(0)=0},y(t),
method=laplace);
MAPLE responds with

$$y(t) = e^{-t} - e^{-t} \cos(t)$$
End of Block 20.4
You should obtain \( Q(s) = \frac{v_0 C}{s(RCs + 1)} \) since, taking the Laplace transform of each term in the differential equation:

\[
R\mathcal{L}\left\{ \frac{dq}{dt} \right\} + \frac{1}{C} \mathcal{L}\{q\} = \mathcal{L}\{v_0\}
\]

i.e.

\[
R[-q(0) + sQ(s)] + \frac{1}{C} Q(s) = \frac{v_0}{s}
\]

where, we emphasize, the Laplace transform of the constant term \( v_0 \) is \( \frac{v_0}{s} \). Inserting \( q(0) = 0 \) we have, after some rearrangement,

\[
Q(s) = \frac{v_0 C}{s(RCs + 1)}
\]
You should obtain \( Q(s) = v_0 C \left[ \frac{1}{s} - \frac{RC}{RCs + 1} \right] \)

Back to the theory
You should obtain \( q(t) = v_0 C(1 - e^{-t/RC})u(t) \) since

\[
\mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} = 1 \quad \text{and} \quad \mathcal{L}^{-1}\left\{ \frac{RC}{RCs + 1} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s + (1/RC)} \right\} = e^{-t/RC}
\]

The solution to this problem is shown in the following diagram.
1. \( x(t) = 3\sin t + 2t \) 
2. \( x(t) = 3e^{2t} \)