Addition and Multiplication Laws of Probability

4.3

Introduction

When we require the probability of two events occurring simultaneously or the probability of one or the other or both of two events occurring then we need probability laws to carry out the calculations.

A short section on Permutations and Combinations is included at the end of this block.

Prerequisites

Before starting this Block you should ...

① understand the ideas of sets and subsets. (Block 4.1)
② understand the concepts of probability and events (Block 4.2)

Learning Outcomes

After completing this Block you should be able to ...

✔ state and use the addition law of probability
✔ understand the term ‘independent events’
✔ state and use the multiplication law of probability
✔ understand the idea of conditional probability

Learning Style

To achieve what is expected of you ...

allocate sufficient study time
briefly revise the prerequisite material
attempt every guided exercise and most of the other exercises
1. The Addition Law

As we have already noted the sample space $S$ is the set of all possible outcomes of a given experiment. Events $A$ and $B$ are subsets of $S$. In the previous block we defined what was meant by $P(A)$, $P(B)$ and their complements in the particular case in which the experiment had equally likely outcomes.

Events, like sets, can be combined to produce new events.

- $A \cup B$ denotes the event that $A$ or $B$ (or both) occur when the experiment is performed.

- $A \cap B$ denotes the event that both $A$ and $B$ occur.

In this block we obtain expressions for determining the probabilities of these combined events written $P(A \cup B)$ and $P(A \cap B)$ respectively.

Experiment 1

A bag contains 18 coloured marbles: 4 are coloured red, 8 are coloured yellow and 6 are coloured green. A marble is selected at random. What is the probability that the ball chosen is either red or green?

Assuming that any marble is as likely to be selected as any other we can say:

- the probability that the chosen marble is red is $\frac{4}{18}$
- the probability that it is green is $\frac{6}{18}$.

It follows that the probability that the ball chosen is either red or green is $\frac{10}{18} = \frac{4}{18} + \frac{6}{18}$. This is the case because no ball can be simultaneously red and green. We say that the events ‘the ball is red’ and ‘the ball is green’ are mutually exclusive.

Experiment 2

Now consider a pack of 52 playing cards. A card is selected at random. What is the probability that the card is either a diamond or a ten?

The probability that it is a diamond is $\frac{13}{52}$ since there are 13 diamond cards in the pack.

The probability that the card is a ten is $\frac{4}{52}$.

There are 16 cards that fall into the category of being either a diamond or a ten: 13 of these are diamonds and there is a ten in each of the three other suits. Therefore, the probability of the card being a diamond or a ten is $\frac{16}{52}$ not $\frac{13}{52} + \frac{4}{52} = \frac{17}{52}$. We say that these events are not mutually exclusive. We must ensure in this case not to simply add the two original probabilities; this would count the ten of diamonds twice - once in each category.
Key Point

The Addition Law of Probability

If two events \( A \) and \( B \) are mutually exclusive then

\[
P(A \cup B) = P(A) + P(B)
\]

This is the simplified version of the Addition Law. However, when \( A \) and \( B \) are not mutually exclusive, \( A \cap B \neq \emptyset \), it can be shown that a more general law applies:

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B)
\]

Of course if \( A \cap B = \emptyset \) then, since \( P(\emptyset) = 0 \) this general expression reduces to the simpler version.

In the last example \( P(A) = \frac{13}{52} \) and \( P(B) = \frac{4}{52} \). The intersection event \( A \cap B \) consists of only one member - the ten of diamonds - hence \( P(A \cap B) = \frac{1}{52} \). Therefore \( P(A \cup B) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} \) as we have already argued.

Now do this exercise

A bag contains 20 marbles, 3 are coloured red, 6 are coloured green, 4 are coloured blue, 2 are coloured white and 5 are coloured yellow. One ball is selected at random. Find the probabilities of the following events.

(a) the ball is either red or green
(b) the ball is not blue
(c) the ball is either red or white or blue. (Hint: consider the complementary event.)

In the last example (part (c)) we could alternatively have used an obvious extension of the law of addition for mutually exclusive events:

\[
P(R \cup W \cup B) = P(R) + P(W) + P(B) = \frac{3}{20} + \frac{2}{20} + \frac{4}{20} = \frac{9}{20}.
\]

Now do this exercise

Figure 1 shows a simplified circuit in which two components \( a \) and \( b \) are connected in parallel.

![Figure 1](image)

The circuit functions if either or both of the components are operational. It is known that if \( A \) is the event ‘component \( a \) is operating’ and \( B \) is the event ‘component \( b \) is operating’ then \( P(A) = 0.99 \), \( P(B) = 0.98 \) and \( P(A \cap B) = 0.9702 \). Find the probability that the circuit is functioning.
More exercises for you to try

1. The following people are in a room: 5 men over 21, 4 men under 21, 6 women over 21, and 3 women under 21. One person is chosen at random. The following events are defined: \( A = \{ \text{the person is over 21} \} \); \( B = \{ \text{the person is under 21} \} \); \( C = \{ \text{the person is male} \} \); \( D = \{ \text{the person is female} \} \). Evaluate the following:

   (a) \( P(B \cap D) \)

   (b) \( P(A' \cup C') \)

   Express the meaning of these quantities in words.

2. A lot consists of 10 good articles, 4 with minor defects, and 2 with major defects. One article is chosen at random. Find the probability that:

   (a) it has no defects

   (b) it has no major defects

   (c) it is either good or has major defects

3. A card is drawn at random from a deck of 52 playing cards. What is the probability that it is an ace or a face card?

4. In a single throw of two dice, what is the probability that neither a double nor a 9 will appear?

Answer

2. Conditional Probability

Suppose a bag contains 6 balls, 3 red and 3 white. Two balls are chosen (without replacement) at random, one after the other. Consider the two events \( A, B \):

   - \( A \) is event “first ball chosen is red”
   - \( B \) is event “second ball chosen is red”

We easily find \( P(A) = \frac{3}{6} = \frac{1}{2} \). However, determining the probability of \( B \) is not quite so straightforward. If the first ball chosen is red then the bag subsequently contains 2 red balls and 3 white. In this case \( P(B) = \frac{2}{5} \). However, if the first ball chosen is white then the bag subsequently contains 3 red balls and 2 white. In this case \( P(B) = \frac{3}{5} \). What this example shows is that the probability that \( B \) occurs is clearly dependent upon whether or not the event \( A \) has occurred. The probability of \( B \) occurring is conditional on the occurrence or otherwise of \( A \).

The conditional probability of an event \( B \) occurring given that event \( A \) has occurred is written \( P(B|A) \). In this particular example

\[
P(B|A) = \frac{2}{5} \quad \text{and} \quad P(B|A') = \frac{3}{5}.
\]

Consider more generally, the performance of an experiment in which the outcome is a member of an event \( A \). We can therefore say that the event \( A \) has occurred.
What is the probability that \( B \) then occurs? That is what is \( P(B|A) \)? In a sense we have a new sample space which is the event \( A \). For \( B \) to occur some of its members must also be members of event \( A \). So \( P(B|A) \) must be the number of outcomes in \( A \cap B \) divided by the number of outcomes in \( A \). That is

\[
P(B|A) = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } A}.
\]

Now if we divide both the top and bottom of this fraction by the total number of outcomes of the experiment we obtain an expression for the conditional probability of \( B \) occurring given that \( A \) has occurred:

**Key Point**

**Conditional Probability**

\[
P(B|A) = \frac{P(A \cap B)}{P(A)} \quad \text{or, equivalently} \quad P(A \cap B) = P(B|A)P(A)
\]

To illustrate the use of conditional probability concepts we return to the example of the bag containing 3 red and 3 white balls in which we consider two events:

- \( A \) is event “first ball is red”
- \( B \) is event “second ball is red”

Let the red balls be numbered 1 to 3 and the white balls 4 to 6. If, for example, (3,5) represents the fact that the first ball is 3 (red) and the second ball is 5 (white) then we see that there are \(6 \times 5 = 30\) possible outcomes to the experiment (no ball can be selected twice.) If the first ball is red then only the fifteen outcomes \((1,x), (2,y), (3,z)\) are then possible (here \(x \neq 1, y \neq 2\) and \(z \neq 3\)). Of these fifteen the six outcomes \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\} will produce the required result, ie the event in which both balls chosen are red, giving a probability: \( P(B|A) = \frac{6}{15} = \frac{2}{5} \).

**Tree diagrams**

A pictorial way of solving such problems is by means of a tree diagram. This is shown in Figure 2.

At the first stage either \( A \) or \( A' \) must occur. If \( A \) is the event ‘the first ball is red’ then \( A' \) is the event ‘the first ball is white’.
Since \( P(A) = \frac{3}{6} = \frac{1}{2} \), then \( P(A') = \frac{1}{2} \), so that the total probability emerging from the left-hand node is 1, shared between the two branches emanating from it.

If \( A \) has occurred (top right branch) then two of the five remaining balls are red so that \( P(B|A) = \frac{2}{5} \). The other option now is \( (B'|A) \) which has probability \( 1 - \frac{2}{5} = \frac{3}{5} \) since the total probability emerging from any node is 1.

Similarly if \( A' \) occurs then there are three red and two white balls left giving \( P(B|A') = \frac{3}{5} \) and therefore \( P(B'|A') = \frac{2}{5} \).

We can use Figure 2 to calculate \( P(B) \) i.e. the probability that the second ball is red, irrespective of what the first ball was. There are two options: \( A \cap B \) and \( A' \cap B \).

To find \( P(A \cap B) \) proceed along the top branches multiplying probabilities as we go so that

\[
P(A \cap B) = P(B|A)P(A) = \frac{2}{5} \times \frac{1}{2} = \frac{1}{5}.
\]

(Remember that 6 of the original 30 events were in \( A \cap B \)).

To find \( P(A' \cap B) \) proceed along the lower branch to \( A' \) and then the upper branch from there

\[
P(A' \cap B) = P(B|A')P(A') = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}.
\]

Now the events \( A \cap B \) and \( A' \cap B \) are clearly mutually exclusive so that

\[
P(B) = \frac{1}{5} + \frac{3}{10} = \frac{5}{10} = \frac{1}{2}.
\]

Here we have used the result \( B \equiv (A \cap B) \cup (A' \cap B) \) and so

\[
P(B) = P((A \cap B) \cup (A' \cap B)) = P(A \cap B) + P(A' \cap B)
\]

A simplified version of a general tree diagram is shown in Figure 3.

Here at the first node (furthest left) there are just two possibilities; either \( A \) or its complement \( A' \) occurs. At the second level of nodes there are again two possibilities; either \( B \) or \( B' \) occurs and so on. In practice there may be many levels and at each node there may be numerous
possibilities. The probability of the event at the end of a branch occurring is obtained by multiplying all the probabilities attached to each separate branch together as the path is traced back to the original node. For example the probability of $A$ and $B'$ occurring is (from the figure)

$$P(A \cap B') = P(B'|A)P(A)$$

**Now do this exercise**

A bag contains 10 balls, 3 red and 7 white. Let $A$ and $B$ be the events as previously. Construct a tree diagram to represent the options, indicating probabilities on the branches.

**Answer**

**Now do this exercise**

Now calculate the probability that the second ball is white.

**Answer**

3. Independent events

If the occurrence of one event $A$ does not affect, nor is affected by, the occurrence of another event $B$ then we say that $A$ and $B$ are independent events. Clearly, if $A$ and $B$ are independent then

$$P(B|A) = P(B) \quad \quad P(A|B) = P(A)$$

Then, using the last keypoint formula we have, for independent events:

<table>
<thead>
<tr>
<th>Key Point</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>The Multiplication Law</strong></td>
</tr>
</tbody>
</table>

If $A$ and $B$ are independent events then

$$P(A \cap B) = P(A)P(B)$$

In words

‘The probability of independent events $A$ and $B$ occurring is the product of the probabilities of the events occurring separately.’

In Figure 4 two components $a$ and $b$ are connected in series.

![Figure 4](image)

Define two events

- $A$ is the event ‘component $a$ is operating’
- $B$ is the event ‘component $b$ is operating’
Previous testing has indicated that $P(A) = 0.99$, and $P(B) = 0.98$. The circuit functions only if $a$ and $b$ are both operating simultaneously. The components are assumed to be independent. Then the probability that the circuit is operating is given by

$$P(A \cap B) = P(A)P(B) = 0.99 \times 0.98 = 0.9702$$

Note that this probability is smaller than either $P(A)$ or $P(B)$.

**More exercises for you to try**

1. A box contains 4 bad and 6 good tubes. Two are drawn out together. One of them is tested and found to be good. What is the probability that the other one is also good?

2. In the above problem the tubes are checked by drawing a tube at random, testing it and repeating the process until all 4 bad tubes are located. What is the probability that the fourth bad tube will be located.
   (a) on the fifth test? (b) on the tenth test?

3. A man owns a house in town and a cottage in the country. In any one year the probability of the house being burgled is 0.01 and the probability of the cottage being burgled is 0.05. In any one year what is the probability that:
   (a) both will be burgled? (b) one or the other (but not both) will be burgled?

4. In a Series, teams $A$ and $B$ play until one team has won 4 games. If team $A$ has probability $\frac{2}{3}$ of winning against $B$ in a single game, what is the probability that the Series will end only after 7 games are played?

5. The probability that a single aircraft engine will fail during flight is $q$. A plane makes a successful flight if at least half its engines run. Assuming that the engines are independent, find the values of $q$ for which a two-engine plane is to be preferred to a four-engined one.

6. Current flows through a relay only if it is closed. The probability of any relay being closed is 0.95. Calculate the probability that a current will flow through a circuit composed of 3 relays in parallel. What assumption must be made?
4. Permutations and Combinations

A permutation is an arrangement. A combination is a group.

Thus \( abc, bac, cab \) are different permutations of the same group \( abc \) of three items \( a, b, \) and \( c \).

In a permutation order is the essential word. In combinations the order of items is of no importance.

Suppose we have \( n \) different items and we wish to find how many different ways they can be arranged, using all of them in each arrangement. This problem is identical with that of seating \( n \) people on \( n \) chairs in every possible way.

The first chair may be filled in \( n \) ways. When this has been done the second chair may be filled in \((n - 1)\) ways since we must leave one person on the first chair. We continue in this way until there is only one person to fill the last chair. Thus the number of ways of arranging \( n \) things using all of them in each arrangement is

\[
 n(n - 1)(n - 2) \ldots 3.2.1 \equiv n!
\]

Finding Permutations

Suppose we have \( n \) items. How many arrangements of \( r \) items can we form from these \( n \) items? The number of arrangements is denoted by \( ^nP_r \). The ‘\( n \)’ refers to the number of different items and the ‘\( r \)’ refers to the number of them appearing in each arrangement. This is equivalent to finding how many different arrangements of people we can get on \( r \) chairs if we have \( n \) people to choose from. We proceed as above.

The first chair can be filled by \( n \) people; the second by \((n - 1)\) people and so on. The \( r^{th} \) chair can be filled by \((n - r + 1)\) people. Hence we easily see that

\[
 ^nP_r = n(n - 1)(n - 2) \ldots (n - r + 1) \equiv \frac{n!}{(n - r)!}
\]

Finding Combinations

The notation \( ^nC_r \) means the number of groups (or combinations) each containing \( r \) things which we can get if we have \( n \) different things from which to choose them.

This problem is identical with that of finding how many different groups of \( r \) people we can get from \( n \) people.

We know that any group of \( r \) people can be arranged in \( r! \) ways. Thus a group of 4 people can be arranged in \( 4! = 24 \) ways.

If we put \( X = ^nC_r \) then \( Xr! \) represents the number of ways in which \( n \) people can form \( r \) arrangements, since every group of \( r \) can arrange themselves in \( r! \) ways. Thus

\[
 Xr! = ^nP_r \quad \text{so} \quad ^nC_r = \frac{^nP_r}{r!} = \frac{n!}{(n - r)!r!}
\]

For example, the number of soccer teams that can be picked from 14 people is

\[
 ^{14}C_{11} = \frac{14!}{3!11!} = \frac{14 \times 13 \times 12}{3 \times 2 \times 1} = 364
\]

However, if teams were regarded as distinct if playing positions mattered then there would be

\[
 ^{14}P_{11} = 14 \times 13 \times 12 \ldots 5.4 \approx 1.4 \times 10^{10}
\]

different teams.
End of Block 4.3
Note that a ball can only have one colour, which are designated by the letters $R, G, B, W, Y$.

(a) \[ P(R \cup G) = P(R) + P(G) = \frac{3}{20} + \frac{6}{20} = \frac{9}{20}. \]

(b) \[ P(B') = 1 - P(B) = 1 - \frac{4}{20} = \frac{16}{20} = \frac{4}{5}. \]

(c) The complementary event is $G \cup Y$ and $P(G \cup Y) = \frac{6}{20} + \frac{5}{20} = \frac{11}{20}$.

Hence \[ P(R \cup W \cup B) = 1 - \frac{11}{20} = \frac{9}{20}. \]
The probability that the circuit is functioning is $P(A \cup B)$. In words: either $a$ or $b$ must be functioning if the circuit is to function. Using the keypoint:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.99 + 0.98 - 0.9702 = 0.9998$$

Not surprisingly the probability that the circuit functions is greater than the probability that either of the individual components functions.
1. (a) \[ P(B \cup D) = P(B) + P(D) - P(B \cap D) \] \[ p(B) = \frac{7}{18}, \quad p(D) = \frac{9}{18} = \frac{1}{2} \]

\[ \therefore P(B \cap D) = \frac{3}{18} = \frac{1}{6} \quad \therefore P(B \cup D) = \frac{7}{18} + \frac{9}{18} - \frac{3}{18} = \frac{13}{18} \]

(b) \[ P(A' \cap C') \quad A' = \{\text{people under 21}\} \quad C' = \{\text{people who are female}\} \]

\[ \therefore P(A' \cap C') = \frac{3}{18} = \frac{1}{6} \]

2. \[ G = \{\text{article is good}\}, \quad M_j = \{\text{major defect}\} \quad M_n = \{\text{minor defect}\} \]

(a) \[ P(G) = \frac{10}{16} = \frac{5}{8} \]

(b) \[ P(G \cup M_n) = P(G) + P(M_n) - P(G \cap M_n) = \frac{5}{8} + \frac{4}{16} - 0 = \frac{7}{8} \]

(c) \[ P(G \cup M_j) = P(G) + P(M_j) - P(G \cap M_j) = \frac{5}{8} + \frac{2}{16} + 0 = \frac{6}{8} = \frac{3}{4} \]

3. \[ F = \{\text{face card}\} \quad A = \{\text{card is ace}\} \quad P(F) = \frac{12}{52}, \quad P(A) = \frac{4}{52} \]

\[ \therefore P(F \cup A) = P(F) + P(A) - P(F \cap A) = \frac{12}{52} + \frac{4}{52} - 0 = \frac{16}{52} \]

4. \[ D = \{\text{double is thrown}\} \quad N = \{\text{sum is 9}\} \]

\[ P(D) = \frac{6}{36} \text{ (36 possible outcomes in an experiment in which all the outcomes are equally probable).} \]

\[ P(N) = P\{(6 \cap 3) \cup (5 \cap 4)(4 \cap 5) \cup (3 \cap 6)\} = \frac{4}{36} \]

\[ P(D \cup N) = p(D) + P(N) - P(D \cap N) = \frac{6}{36} + \frac{4}{36} - 0 = \frac{10}{36} \]

\[ P(D \cup N) = 1 - P(D \cup N) = 1 - \frac{10}{36} = \frac{13}{18} \]
Back to the theory
The two mutually exclusive events are \( A \cap B' \) and \( A' \cap B' \). However the event that the second ball chosen is white is \( B' \) and \( B' = (A \cap B') \cup (A' \cap B') \)

\[
P(A \cap B') = P(B'|A)P(A) = \frac{7}{9} \times \frac{3}{10} = \frac{21}{90}
\]

\[
P(A' \cap B') = P(B'|A')P(A') = \frac{6}{9} \times \frac{7}{10} = \frac{42}{90}
\]

Therefore the required probability is \( \frac{21}{90} + \frac{42}{90} = \frac{63}{90} = \frac{7}{10} \).
1. Let $G_i = \{i^{th} \text{ tube is good}\}$, $B_i = \{i^{th} \text{ tube is bad}\}$

\[ P(G_2|G_1) = \frac{5}{9} \] (only 5 good tubes left out of 9).

2. Same events as in question 1.

(a) This will occur if event

\[ \{B_1 \cap B_2 \cap B_3 \cap G_4 \cap B_5\} \cup \{\ldots \cap B_3\} \cup \{\ldots B_3\} \cup \{\ldots B_5\} \] occurs.

Here we have a number of events in which $B_5$ must appear in the last position and there must be just three appearances of the $B$ symbol in the first 4 slots. Now the number of ways of arranging 3 from 4 is 4C3 (see following Section). Thus the probability of the required event occurring is $4C_3 P\{B_1 \cap B_2 \cap B_3 \cap G_4 \cap B_5\}$.

\[
\begin{align*}
\therefore \quad P(\text{event occurring}) &= 4C_3 P\{B_1 \cap B_2 \cap B_3 \cap G_4 \cap B_5\} \\
&= 4C_3 P(B_5)(B_1 \cap B_2 \cap B_3 \cap G_4) P(B_1 \cap B_2 \cap B_3 \cap G_4) \\
&= 4C_3 P(B_5)(B_1 \cap B_2 \cap B_3 \cap G_4) P(B_1)P(B_2)P(B_3) \\
&= 4C_3 \frac{1}{6} \cdot \frac{6}{7} \cdot \frac{2}{8} \cdot \frac{3}{9} \cdot \frac{4}{10} = \frac{4}{210} = \frac{2}{105}
\end{align*}
\]

(b) Same idea as in (a)

Req’d probability = $4C_3 P\{B_{10}|B_1 \cap B_2 \ldots \cap G_4 \cap \ldots \cap G_9\} \cdot P\{B_1 \cap \ldots \cap G_9\}$

\[
= 4C_3 \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdot \frac{2}{8} \cdot \frac{3}{9} \cdot \frac{4}{10} = \frac{2}{5}
\]

3. $H = \{\text{house is burgled}\}$, $C = \{\text{cottage is burgled}\}$

$P(H \cap C) = P(H)P(C) = (0.01)(0.05) = 0.0005$ since events independent

$P(\text{one or the other (but not both)}) = P((H \cap C) \cup (H' \cap C)) = P(H \cap C') + P(H' \cap C)$

\[
= P(H)P(C') + P(H')P(C)
\]

\[
= (0.01)(0.95) + (0.99)(0.05) = 0.059.
\]

4. Let $A_i$ be event $\{A \text{ wins the } i^{th} \text{ game}\}$

req’d event is $\{A_1 \cap A_2 \cap A_3 \cap A_4' \cap A_5' \cap A_6'\} (\ldots)(\ldots)$

no. of ways of arranging 3 in 6 i.e. 6C3

\[
P(\text{req’d event}) = 6C_3 P(A_1 \cap A_2 \cap A_3 \cap A_4' \cap A_5' \cap A_6') = 6C_3 [P(A_1)]^3 P[A_1']^3 = \frac{160}{729}
\]
5. Let $E_i$ be event \{\text{$i^{th}$ engine success}\}

2 engine plane: flight success if \{(E_1 \cap E_2) \cup (E'_1 \cap E_2)(E_1 \cap E'_2)\} occurs

$$P(\text{req'd event}) = P(E_1)P(E_2) + P(E'_1)P(E_2) + P(E_1)P(E'_2) = (1-q)^2 + 2q(1-q) = 1 - q^2$$

4 engine plane: success if following event occurs

$$\begin{align*}
\binom{4}{2} \text{ ways} & \{E_1 \cap E_2 \cap E'_3 \cap E'_4\} \cap \{E_1 \cap E_2 \cap E_3 \cap E'_4\} \cap \{E_1 \cap E_2 \cap E_3 \cap E_4\} \\
\binom{4}{1} \text{ ways} & \{E_1 \cap E_2 \cap E_3 \cap E'_4\} \cap \{E_1 \cap E_2 \cap E'_3 \cap E_4\} \\
\binom{4}{0} \text{ ways} & \{E_1 \cap E_2 \cap E_3 \cap E_4\}
\end{align*}$$

req’d probability = $6(1-q)^2q^2 + 4(1-q)^3q + (1-q)^4 = 3q^4 - 4q^3 + 1$

Two engine plane is preferred if

$$1 - q^2 > 3q^4 - 4q^3 + 1 \quad \text{i.e. if} \quad 0 > q^2(3q - 1)(q - 1)$$

Let $y = (3q - 1)(q - 1)$. By drawing a graph of this quadratic you will quickly see that a two engine plane is preferred if $\frac{1}{3} < q < 1$

6. Let $A$ be event \{relay $A$ is closed\}: Similarly for $B$, $C$

req’d event is \{$A \cap B \cap C$} \cup \{$A' \cap B \cap C$} \cap \{$A' \cap B' \cap C$} \cap \{$A \cap B' \cap C$} \cap \{$A' \cap B \cap C$} \cap \{$A' \cap B' \cap C$}

$$P(\text{req'd event}) = (0.95)^3 + 3(0.95)(0.05) + 3(0.95)(0.05)^2 = 0.999875$$

( or $1 - p(\text{all relays open}) = 1 - (0.05)^3 = 0.999875.$)

Back to the theory