The Hyperbolic Functions

6.2

Introduction

The hyperbolic functions \( \cosh x \), \( \sinh x \), \( \tanh x \) etc are certain combinations of the exponential functions \( e^x \) and \( e^{-x} \). The notation implies a close relationship between these functions and the trigonometric functions \( \cos x \), \( \sin x \), \( \tan x \) etc. The close relationship is algebraic rather than geometrical. For example, the functions \( \cosh x \) and \( \sinh x \) satisfy the relation

\[
\cosh^2 x - \sinh^2 x = 1
\]

which is very similar to the trigonometric identity \( \cos^2 x + \sin^2 x = 1 \). (In fact any trigonometric identity has an equivalent hyperbolic function identity).

The hyperbolic functions are not introduced because they are a mathematical nicety. These combinations of exponentials do arise naturally and sufficiently often to warrant sustained study. For example, the shape of a chain hanging under gravity is well described by \( \cosh x \) and the deformation of uniform beams can be expressed in terms of hyperbolic tangents.

Prerequisites

Before starting this Block you should...

1. have a good knowledge of the exponential function
2. have knowledge of odd and even functions
3. have familiarity with the definitions of \( \tan x \), \( \sec x \), \( \cosec x \) and of trigonometric identities

Learning Outcomes

After completing this Block you should be able to...

1. understand how hyperbolic functions are defined in terms of exponential functions
2. be able to obtain hyperbolic function identities and manipulate expressions involving hyperbolic functions

Learning Style

To achieve what is expected of you...

- allocate sufficient study time
- briefly revise the prerequisite material
- attempt every guided exercise and most of the other exercises
1. Constructing even and odd functions

A given function \( f(x) \) can always be split into two parts, one of which is even and one of which is odd. To do this write \( f(x) \) as \( \frac{1}{2}[f(x) + f(-x)] \) and then simply add and subtract \( \frac{1}{2}f(-x) \) to this to give

\[
f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]
\]

The term \( \frac{1}{2}[f(x) + f(-x)] \) is even because when \( x \) is replaced by \( -x \) we have \( \frac{1}{2}[f(-x) + f(x)] \) which is the same as the original. However, the term \( \frac{1}{2}[f(x) - f(-x)] \) is odd since, on replacing \( x \) by \( -x \) we have \( \frac{1}{2}[f(-x) - f(x)] = -\frac{1}{2}[f(x) - f(-x)] \) which is the negative of the original.

**Try each part of this exercise**

Separate the function \( x^2 - 3x \) into odd and even parts.

Part (a) First, define \( f(x) \) and find \( f(-x) \).

Part (b) Now construct \( \frac{1}{2}[f(x) + f(-x)], \frac{1}{2}[f(x) - f(-x)] \)

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**The odd and even parts of the exponential function**

Using the approach outlined above we see that the even part of \( e^x \) is

\[
\frac{1}{2}(e^x + e^{-x})
\]

and the odd part of \( e^x \) is

\[
\frac{1}{2}(e^x - e^{-x})
\]

We give these new functions special names: \( \cosh x \) (pronounced ‘cosh’ \( x \)) and \( \sinh x \) (pronounced ‘shine’ \( x \))

<table>
<thead>
<tr>
<th>Key Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cosh x = \frac{1}{2}(e^x + e^{-x}) ) \quad \sinh x = \frac{1}{2}(e^x - e^{-x}) )</td>
</tr>
</tbody>
</table>

\( \cosh x \) and \( \sinh x \) are called **hyperbolic functions**

These two relations, when added and subtracted, give

\[
e^x = \cosh x + \sinh x \quad \text{and} \quad e^{-x} = \cosh x - \sinh x
\]

The hyperbolic functions are closely related to the trigonometric functions \( \cos x \) and \( \sin x \). Indeed, this explains the notation that we use. The hyperbolic cosine is written ‘\( \cos \)’ with a ‘\( h \)’
to get cosh and the hyperbolic sine is written ‘sin’ with a ‘h’ to get sinh. The graphs of cosh \(x\) and sinh \(x\) are shown in the following diagram.

Note that \(\cosh x > 0\) for all values of \(x\) and that \(\sinh x\) only vanishes when \(x = 0\).

2. Hyperbolic identities

The hyperbolic functions \(\cosh x\), \(\sinh x\) satisfy similar (but not identical) identities to those satisfied by \(\cos x\), \(\sin x\). We note first, some basic notation similar to that employed with trigonometric functions:

\[
\cosh^n x \text{ means }(\cosh x)^n \quad \sinh^n x \text{ means } (\sinh x)^n \quad n \neq -1
\]

In the special case that \(n = -1\) we do not use \(\cosh^{-1} x\) and \(\sinh^{-1} x\) to mean \(\frac{1}{\cosh x}\) and \(\frac{1}{\sinh x}\) respectively. (The notation \(\cosh^{-1} x\) and \(\sinh^{-1} x\) is reserved for the inverse functions of \(\cosh x\) and \(\sinh x\) respectively).

**Try each part of this exercise**

Show that \(\cosh^2 x – \sinh^2 x = 1\) for all \(x\).

Part (a) First find an expression for \(\cosh^2 x\) in terms of the exponential functions \(e^x\), \(e^{-x}\)

\[
\cosh^2 x = \left[\frac{1}{2}(e^x + e^{-x})\right]^2 = \quad \text{Answer}
\]

Part (b) Similarly, find an expression for \(\sinh^2 x\) in terms of \(e^x\), \(e^{-x}\)

\[
\sinh^2 x = \left[\frac{1}{2}(e^x - e^{-x})\right]^2 = \quad \text{Answer}
\]

Part (c) Finally determine \(\cosh^2 x – \sinh^2 x\).

\[
\cosh^2 x – \sinh^2 x = \frac{1}{4}[e^{2x} + 2 + e^{-2x}] - \frac{1}{4}[e^{2x} - 2 + e^{-2x}] = \quad \text{Answer}
\]

As an alternative to the calculation in this guided exercise we could, instead, use the relations

\[
e^x = \cosh x + \sinh x \quad e^{-x} = \cosh x - \sinh x
\]
and so, remembering the algebraic identity: \((a + b)(a - b) = a^2 - b^2\) we see that
\[(\cosh x + \sinh x)(\cosh x - \sinh x) = e^x e^{-x} = 1\]
that is \(\cosh^2 x - \sinh^2 x = 1\)

**Key Point**

The fundamental identity relating hyperbolic functions is:

\[\cosh^2 x - \sinh^2 x = 1\]

This is the hyperbolic function equivalent of the trigonometric identity: \(\cos^2 x + \sin^2 x = 1\)

**Try each part of this exercise**

Show that \(\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y\).

**Part (a) First, find \(\cosh x \cosh y\) in terms of exponentials.**

\[
cosh x \cosh y = \left(\frac{e^x + e^{-x}}{2}\right) \left(\frac{e^y + e^{-y}}{2}\right) = \]

[Answer]

**Part (b) Now find \(\sinh x \sinh y\)**

\[
\sinh x \sinh y = \left(\frac{e^x - e^{-x}}{2}\right) \left(\frac{e^y - e^{-y}}{2}\right) = \]

[Answer]

**Part (c) Now find \(\cosh x \cosh y + \sinh x \sinh y\) and express the result in terms of a hyperbolic function.**

[Answer]

Other hyperbolic function identities can be found in a similar way. The most commonly used hyperbolic identities are listed in the following keypoint.

**Key Point**

- \(\cosh^2 - \sinh^2 = 1\)
- \(\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y\)
- \(\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y\)
- \(\sinh 2x = 2 \sinh x \cosh y\)
- \(\cosh 2x = \cosh^2 x + \sinh^2 x\) or \(\cosh 2x = 2 \cosh^2 - 1\) or \(\cosh 2x = 1 + 2 \sinh^2 x\)
3. Related hyperbolic functions

Once the trigonometric functions \( \cos x, \sin x \) are introduced then related functions are also introduced; \( \tan x, \sec x, \cosec x \) through the relations:

\[
\tan x = \frac{\sin x}{\cos x} \quad \sec x = \frac{1}{\cos x} \quad \cosec x = \frac{1}{\sin x}
\]

In an exactly similar way we introduce hyperbolic functions \( \tanh x, \sech x \) and \( \cosech x \) (again the notation is obvious: take the ‘trigonometric’ name and append the letter ‘h’). These functions are defined in the following keypoint

**Key Point**

<table>
<thead>
<tr>
<th>Function</th>
<th>Definition</th>
</tr>
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<tbody>
<tr>
<td>( \tanh x )</td>
<td>( \frac{\sinh x}{\cosh x} )</td>
</tr>
<tr>
<td>( \sech x )</td>
<td>( \frac{1}{\cosh x} )</td>
</tr>
<tr>
<td>( \cosech x )</td>
<td>( \frac{1}{\sinh x} )</td>
</tr>
</tbody>
</table>

**Try each part of this exercise**

Show that

(a) \( \sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y \)

(b) \( 1 - \tanh^2 x = \sech^2 x \)

Part (a)(i) Use the identity \( \sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y \) and replace \( y \) by \( -y \).

\( \sinh(x - y) = \)

Part (a)(ii) Now obtain expressions for \( \cosh(-y) \) and \( \sinh(-y) \).

\( \cosh(-y) = \quad \sinh(-y) = \)

Part (a)(iii) Now complete the problem

\( \sinh(x - y) = \sinh x \cosh(-y) + \cosh x \sinh(-y) = \)

Part (b) Use the identity \( \cosh^2 x - \sinh^2 x = 1 \).

\( \cosh^2 x - \sinh^2 x = 1 \) so

**More exercises for you to try**

1. Express
   (a) \( 2\sinh x + 3\cosh x \) in terms of \( e^x \) and \( e^{-x} \).
   (b) \( 2\sinh 4x - 7\cosh 4x \) in terms of \( e^{4x} \) and \( e^{-4x} \).

2. Express
   (a) \( 2e^x - e^{-x} \) in terms of \( \sinh x \) and \( \cosh x \).
   (b) \( \frac{7e^x}{(e^x - e^{-x})} \) in terms of \( \sinh x \) and \( \cosh x \), and then in terms of \( \coth x \).
   (c) \( 4e^{-3x} - 3e^{3x} \) in terms of \( \sinh 3x \) and \( \cosh 3x \).

3. Using only the \( \cosh \) and \( \sinh \) keys on your calculator find the values of
   (a) \( \tanh 0.35 \),  (b) \( \cosech 2 \),  (c) \( \sech(0.6) \).

**Answer**
4. Computer Exercise or Activity

For this exercise it will be necessary for you to access the computer package DERIVE.

The hyperbolic functions sinh, cosh, tanh, sech are available in Derive and cosech is available but is denoted by csch. Also coth is available (this function, not referred to in the block is 1/tanh). When DERIVE does any calculations with the hyperbolic functions it reverts to the definitions of these functions in terms of exponentials.
End of Block 6.2
\[ f(x) = x^2 - 3^x, \quad f(-x) = x^2 - 3^{-x} \]
\[ \frac{1}{2}[f(x) + f(-x)] = \frac{1}{2}(x^2 - 3x + x^2 - 3^{-x}) = x^2 - \frac{1}{2}(3^x + 3^{-x}). \] This is the even part of \( f(x) \).

\[ \frac{1}{2}[f(x) - f(-x)] = \frac{1}{2}(x^2 - 3^x - x^2 + 3^{-x}) = \frac{1}{2}(3^{-x} - 3^x). \] This is the odd part of \( f(x) \).

Back to the theory.
\[
\cosh^2 x = \frac{1}{4}(e^x + e^{-x})^2 = \frac{1}{4}[(e^x)^2 + 2e^x e^{-x} + (e^{-x})^2] = \frac{1}{4}[e^{2x} + 2e^x e^{-x} + e^{-2x}] = \frac{1}{4}[e^{2x} + 2 + e^{-2x}]
\]
\[ \sinh^2 x = \frac{1}{4} (e^x - e^{-x})^2 = \frac{1}{4} [(e^x)^2 - 2e^xe^{-x} + (e^{-x})^2] = \frac{1}{4} [e^{2x} - 2e^x - 2e^{-x} + 2e^{-2x}] = \frac{1}{4} [e^{2x} - 2e^{-2x}] \]
\[ \cosh^2 x - \sinh^2 x = 1 \]
\[ \cosh x \cosh y = \left( \frac{e^x + e^{-x}}{2} \right) \left( \frac{e^y + e^{-y}}{2} \right) = \frac{1}{4} [e^x e^y + e^{-x} e^y + e^x e^{-y} + e^{-x} e^{-y}] \]
\[ = \frac{1}{4} (e^{x+y} + e^{-x+y} + e^{x-y} + e^{-x-y}) \]
\[
\sinh x \sinh y = \left( \frac{e^x - e^{-x}}{2} \right) \left( \frac{e^y - e^{-y}}{2} \right) = \frac{1}{4} (e^{x+y} - e^{-x+y} - e^{x-y} + e^{-x-y})
\]
cosh \ x \ cosh \ y + \ sinh \ x \ sinh \ y = \frac{1}{2}(e^{x+y} + e^{-(x+y)}) \text{ which we recognise as } \cosh(x + y)
\[ \sinh(x - y) = \sinh x \cosh(-y) + \cosh x \sinh(-y). \]
\( \cosh(-y) = \cosh y \) since \( \cosh \) is even. Also \( \sinh(-y) = -\sinh y \) since \( \sinh \) is odd.

Back to the theory.
\[ \sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y \]
Dividing both sides by $\cosh^2 x$ gives

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

implying (see last keypoint)

$$1 - \tanh^2 x = \text{sech}^2 x$$

Back to the theory
1. (a) \( \frac{5}{2}e^x - \frac{1}{2}e^{-x} \)  
   (b) \(-\frac{5}{2}e^{4x} - \frac{9}{2}e^{-4x}\)

2. (a) \( \cosh x + 3 \sinh x \)  
   (b) \( \frac{7(\cosh x + \sinh x)}{2 \sinh x} \)  
   (c) \( \frac{7}{2}(\coth x + 1) \)  
   (c) \( \cosh 3x - 7 \sinh 3x \)

3. (a) 0.3364  
   (b) 0.2757  
   (c) 0.8436

Back to the theory.